1. **(10 points)** Let $G = (V, E)$ be an undirected graph without isolated vertices. (Let $n = |V|$ and $m = |E|$.) Let ℓ be the size of a maximum matching in G. Let c be the maximum number of connected components in any subgraph of G; in this definition, a subgraph of G is one whose edges are any subset E' of E, and whose vertices are only the vertices in V which are incident on some edge in E'. Let k be the size of a minimum edge cover of G: an edge cover is a set $E'' \subseteq E$ such that every vertex of V is incident on some edge in E''.

   Prove that $\ell = c = n - k$.

2. **(10 points)** Let $G = (V, E)$ be an undirected graph, and let $V = \{1, ..., n\}$. The Tutte matrix of G, $T(G)$, is the $n \times n$ matrix with

$$T(G)_{u,v} = \begin{cases} x_{u,v} & \text{if } (u, v) \in E \text{ and } u < v \\ -x_{v,u} & \text{if } (u, v) \in E \text{ and } u > v \\ 0 & \text{otherwise} \end{cases}$$

   Here the $\{x_{u,v}\}$ are variables, and the determinant $\det(T(g))$ is a multivariate polynomial in these variables.

   Show that G has a perfect matching if and only if $\det(T(g)) \neq 0$ as a polynomial.

3. **(15 points)** Let J be a set of jobs to be executed, on two processors, with all jobs requiring 1 unit of time. Suppose that there is a directed acyclic "dependency graph" among the jobs, $P = (J, A)$, such that if $(J_1, J_2) \in A$, then J_1 must be executed before J_2. A legal schedule is a function $s : J \to \{1, ..., T\}$ satisfying the conditions

   (i) $|s^{-1}(i)| \leq 2 \ \forall i \in \{1, ..., T\}$.

   (ii) If $(J_1, J_2) \in A$ then $s(J_1) < s(J_2)$.

   The scheduling problem is to find an optimal schedule, namely a legal schedule which minimizes T.

   (a) Define the graph $G_P = (J, E)$ where $(J_1, J_2) \in E$ if there is no path from J_1 to J_2 or from J_2 to J_1 in P. Suppose that M is a maximum matching of G_P. Show that for any legal schedule, $T \geq |J| - |M|$.

   (b) Show that there is a legal schedule satisfying $T = |J| - |M|$.

   (c) Give an efficient algorithm to find an optimal schedule.

4. **(5 points)** Given an undirected graph $G = (V, E)$, a vertex $v \in V$, and an integer k, we wish to find whether there is a spanning tree of G in which v has degree k or less. Show that this is a matroid intersection problem, i.e., a problem of computing a maximum size set in the intersection of two matroids.

5. **(5 points)** Given a bipartite graph $G = (U, V, E)$, let F_U consist of all subsets S of E satisfying the following condition: if $u_1, u_2 \in U, v_1, v_2 \in V$, and $(u_1, v_1), (u_2, v_2) \in S$, then $u_1 \neq u_2$. Similarly let F_V consist of all subsets S of E satisfying the following condition: if $u_1, u_2 \in U, v_1, v_2 \in V$, and $(u_1, v_1), (u_2, v_2) \in S$, then $v_1 \neq v_2$.

   Show that each of F_U and F_V is a matroid. Now suppose that a weight function $w : E \to \mathbb{R}_+$ is given. Formulate the maximum weight bipartite matching problem as that of finding the maximum weight set in the intersection of two matroids. Describe how an augmenting path preserves membership in each of the matroids.

6. **(10 points)** Given an undirected graph $G = (V, E)$, let F denote the collection of subgraphs of G in which every vertex has degree at most k. Show for all k that F is generally not a matroid. (Of course, there are some graphs for which it is.)