1. **(10 points)** n gentlemen arrive at a club and leave their hats with the hat-check boy. The latter, not finding his job terribly interesting, seeks refuge in the bottle. Upon leaving the club the gentlemen each receive a random (uniformly chosen) hat. How many men, on average, get their own hat back?

   Use the Markov inequality to give an upper bound on the probability that at least half get their own hat back. Do you think this bound is tight?

2. **(15 points)** Recall that for prime p, $\mathbb{Z}/p$ is a finite *field*. (It is often denoted $GF(p)$, the "Galois field of size p".) We can, of course, define polynomials over this field: if $a_k, ..., a_0 \in GF(p)$ then for $x \in GF(p)$ the polynomial $f(x) = a_k x^k + ... + a_0$ takes on a value in $GF(p)$.

   Assume that $k = 3$ and $p > k$. Consider the sample space consisting of the uniform distribution on the set of degree 3 polynomials $S = \{a_3x^3 + a_2x^2 + a_1x + a_0\}$.

   For each $b \in GF(q)$, define the random variable X_b as follows: if the polynomial $a_3x^3 + ... + a_0$ is chosen, then $X_b = a_3b^3 + ... + a_0$.

   Show that the rvs are *uniform and* 4*-wise independent*, meaning that for every distinct $b_1, b_2, b_3, b_4 \in GF(p)$, there is an equal probability that the rvs $X_{b_1}, X_{b_2}, X_{b_3}, X_{b_4}$ take on any list of values $X_{b_1} = c_1, X_{b_2} = c_2, X_{b_3} = c_3, X_{b_4} = c_4$. (So of course this probability is simply p^{-4}.)

3. **(10 points)** You've been invited to play on the TV game show "You've got a hankering to be a Millionare", where the host holds out two envelopes and asks you to pick one; each contains a check written out in your name. You pick one of them and take a peek inside; the check is for \$ x = \$ 1000. The host then explains to you that one of the checks was written for a sum twice as large as the other, and he asks you if you want to switch envelopes! He points out that you were equally likely to have chosen the large or small check when you picked an envelope; if you chose the small one, then you'll be switching from \$ x to \$ $2x$, while if you chose the large one you'll be switching from \$ x to \$ $x/2$, and the expected value of your winnings after the switch is therefore \$ $5x/4$.

   Is it better to switch? Explain.