Homework # 5

Due Tuesday, February 27, 2001, at 2:00 PM

1. **Weight Size and Boolean Symbols**
   In this problem we explore how the size of the weights is affected by the choice of the representation of the Boolean symbols. Namely, by choosing whether a threshold function is implemented over $\{0, 1\}$ or over $\{-1, 1\}$. We assume that the weights are integers.

   (a) Let $f(x_1, x_2, x_3) = sgn(-4 + x_1 + 2x_2 + 5x_3)$ for $x_i \in \{0, 1\}$. Compute $S(w) = \sum_{i=0}^{3} |w_i|$ where w_i are the weights. Find new weights w'_i, (implementing the same function) such that $S(w') < S(w)$.

   (b) Given a threshold function f, let W be the set of all possible weight vectors that implement f. We define the *weight size* of f as :

   $$\hat{S}(f) = \min_{W} \sum_{i=0}^{n} |w_i|$$

   That is, the sum of the absolute values of the smallest possible weights that implement the function f.

   We saw in class that $OR(x_1, ..., x_n) = sgn(-1+x_1+...+x_n)$, for $x_i \in \{0, 1\}$. Prove that those weights are minimal. That is, for any other weights, w'_i, that implement the OR function we have $S(w'_i) \geq n + 1$. Namely, $\hat{S}(OR) = n + 1$ over $\{0, 1\}$.

   (c) Find $\hat{S}(OR)$ over $\{-1, 1\}$.

   (d) We define the majority function, MAJ, of an odd number of variables n, as :

   $$MAJ(x_1, ..., x_n) = \begin{cases} 1 & \text{if more than half of the inputs are 1} \\ 0 & \text{otherwise} \end{cases}$$

   Show that MAJ is a threshold function by finding a set of weights that implements it over $\{0, 1\}$.

(e) Find $\hat{S}(MAJ)$ over $\{0, 1\}$.

(f) Find $\hat{S}(MAJ)$ over $\{-1, 1\}$.

(g) Prove/disprove that for an arbitrary threshold function f, $\hat{S}(f)$ over $\{0, 1\}$ is always smaller than $\hat{S}(f)$ over $\{-1, 1\}$.

2. **Batcher Sorting Network**

A comparator is a device which sorts two numbers x and y as shown in Figure 1.

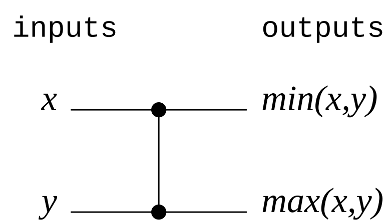

Figure 1: a comparator.

The Batcher sorting network was presented in class. It is constructed with a block called the `Merge` network. Given two sorted input sequences $x_0, x_1, \ldots, x_{n/2-1}$ and $x'_0, x'_1, \ldots, x'_{n/2-1}$, the `Merge[`$n$`]` network produces a sorted output sequence $y_0, y_1, \ldots, y_{n-1}$. The recursive construction of the `Merge[8]` network is shown in Figure 2. The recursive construction of the `Batcher[8]` network, based on the `Merge[8]` network, is shown in Figure 3. The `Merge[2]` and `Batcher[2]` networks both consist of a single balancer.

(a) What is the depth of the `Merge[`n`]` network, $n \geq 2$?

(b) What is the depth of the `Batcher[`n`]` network, $n \geq 2$?

(c) Note that the `Batcher[4]` network consists of six comparators, as shown in Figure 4. Find a sorting network with four inputs consisting of only five comparators. (**Hint:** Try two comparators in the first stage, two in the second stage and one in the third stage.) Prove that your network sorts all possible input sequences.

3. **Smoothing Networks**

A balancer is a device which balances two integers x and y as shown in Figure 5.

The Batcher counting network presented in class has depth $O(\log^2 n)$. In this problem, we consider a network of balancers with depth $\log n$.†

†All logarithms are base 2.

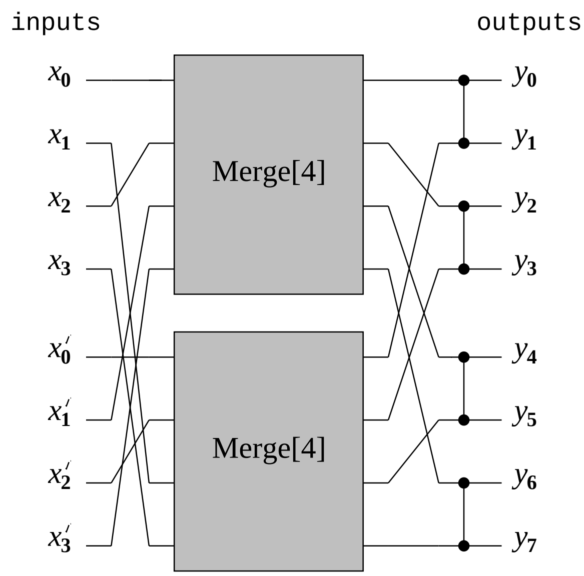

Figure 2: the `Merge[8]` network.

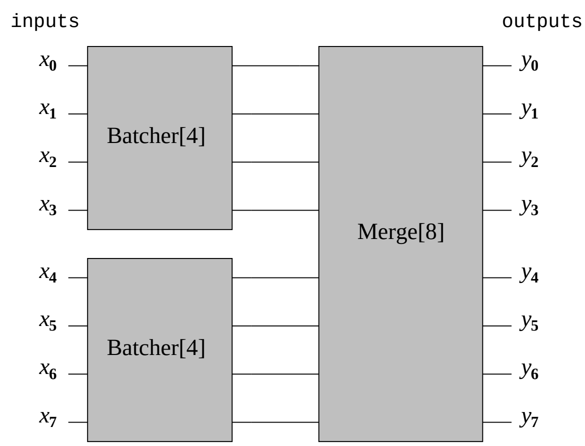

Figure 3: the `Batcher[8]` network.

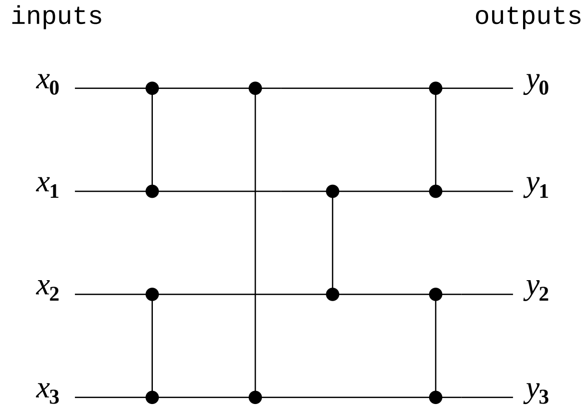

Figure 4: the `Batcher[4]` network.

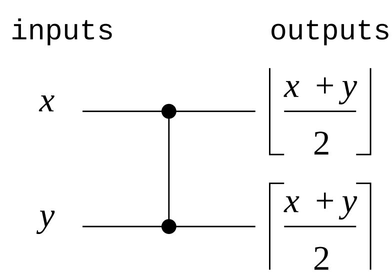

Figure 5: a balancer.

Consider the network shown in Figure 6.

(a) If the inputs x_0, x_1, x_2, and x_3 are 7, 3, 2 and 6, respectively, what are the outputs y_0, y_1, y_2, and y_3?

(b) For an arbitrary input sequence, prove that y_0 is the smallest output value and y_3 is the largest. Prove that the maximum difference between y_0 and y_3 is two.

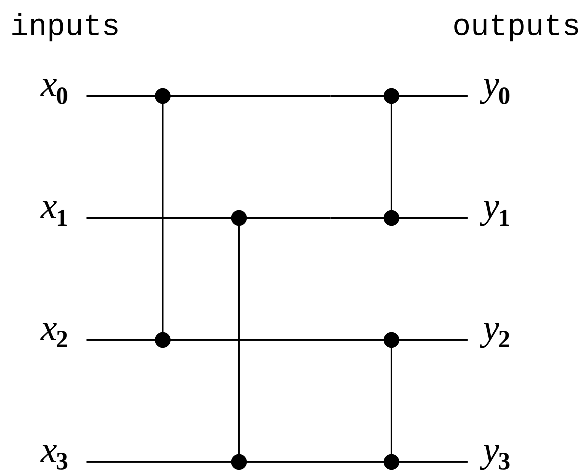

Figure 6: the `Smooth[4]` network.

Consider the `Smooth[`n`]` network, $n \geq 2$, shown in Figure 7. It is constructed recursively with two `Smooth[`$n/2$`]` networks and $n/2$ balancers. The `Smooth[2]` network consists of a single balancer.

(c) Draw the `Smooth[8]` network. Prove that the maximum difference between y_0 and y_7 is three.

(d) For the `Smooth[`n`]` network, $n \geq 2$, prove that the maximum difference between y_0 and y_{n-1} is $\log n$.

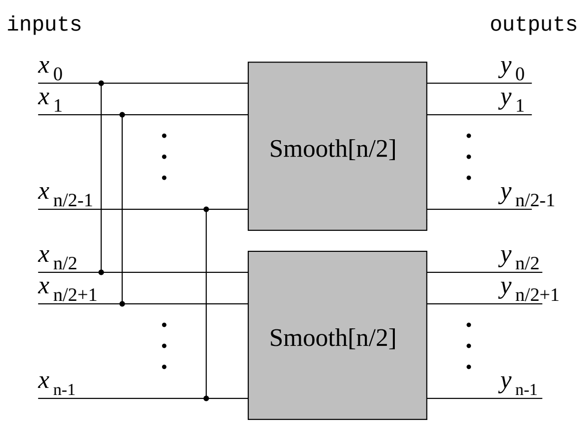

Figure 7: the `Smooth[`n`]` network.