Homework # 3

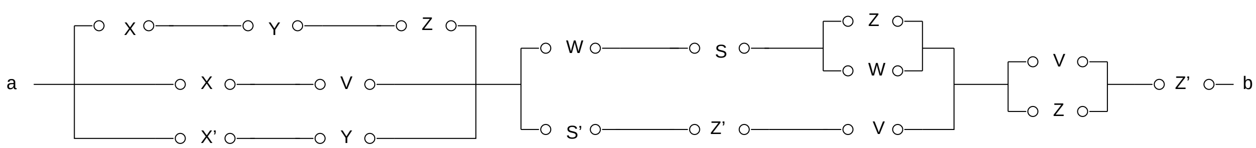
Due Tuesday, February 13, 2001, at 2:00 PM

Collaboration and/or discussion are not allowed on this homework set

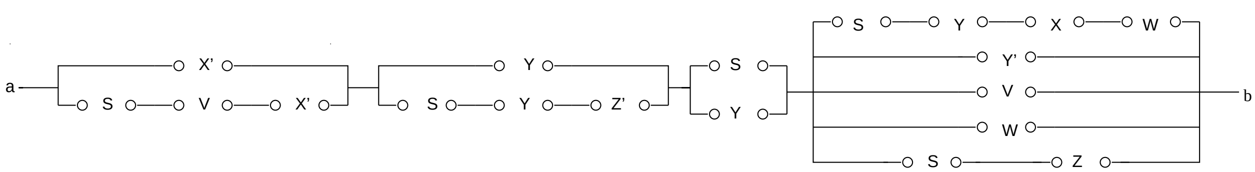
The first three questions are based on C. E. Shannon's Master Thesis from 1938, "A Symbolic Analysis of Relay and Switching Circuits". This classical paper was the first work that connected between logic and circuit design.

1. **Simplification of Relay Circuits**
   Using De Morgan's Theorem and the theorems given on page 476 of Shannon's Masters Thesis, simplify the following circuits as much as possible and draw the simplified circuits. The number of relays expected in the final answer are given for you.

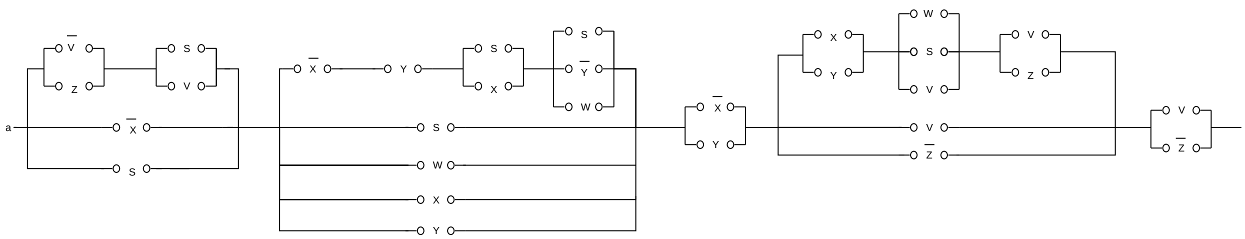
   (a) This circuit can be simplified to a total of at most 6 relays.



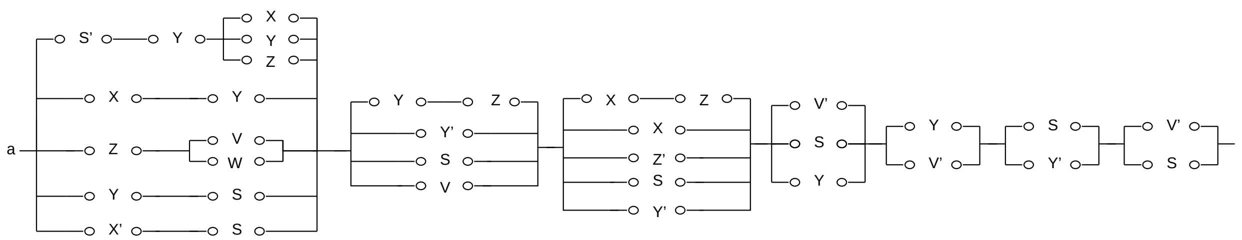
   (b) This circuit can be simplified to a total of at most 6 relays.



(c) This circuit can be simplified to a total of at most 9 relays.



(d) This circuit can be simplified to a total of at most 9 relays.



2. **Boolean Functions and Switching Circuits**
   In Shannon's Master Thesis, he gives a general construction for symmetric functions using only switching elements. He then gives basic rules for simplifying the size of these circuits.

   (a) Find the smallest number of elements, according to Shannon's rules, required to implement $XOR(x_1, ..., x_n)$.

   (b) For any $f(x_1, ..., x_n) \in TH$, find the smallest number of elements required to implement f, as a function of n and i, the threshold level of f.

   (c) Describe a general construction for an arbitrary LT function, using only switching elements. How many elements does your circuit require, as a function of the weight values? (You may assume that the function has only integer weights.)

3. **Some Size Bounds on Switching Circuits**
   In Shannon's Master Thesis, he explains how to develop a series expansion of an arbitrary boolean function as either a sum of products or a product of sums.

   (a) Assume that we can show that any function of n variables requires no more than g_n switching elements. Show that any function of $n+1$ variables can be implemented in at most $2g_n + 2$ elements.

(b) Starting with $g_0 = 0$, use your method from part (a) to show that any boolean function can be implemented in a switching circuit with at most $\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$ elements. This just corresponds to the full expansion of f around each of its variables which is just the minterm or maxterm representation of f.

(c) By expanding around all but one of the variables of a function, we get the partial minterms of the first $n-1$ variables, each coupled to a function of one variable. Since only two switching elements are required to implement all possible functions of one variable, show that any boolean function can be implemented with $2 + \sum_{i=1}^{n-1} 2^i = 2^n$ elements.

(d) For larger n, it is possible to find a better bound by expanding about all but two variables.

- Construct a circuit using 20 switches that computes all 16 functions of two variables.
- Use this circuit to show that any function of $n > 2$ variables can be constructed using no more than $2^{n-1} + 18$ elements.

(e) Show that it is possible to construct a circuit using no more than $2 \times 2^{2^m}$ switches that computes all 2^{2^m} functions of m variables, and use this to show that any function of $n > 2$ variables can be constructed using no more than $2^{n-m+1} + 2 \times (2^{2^m} - 1)$ switches, for any choice of m. What is the smallest n for which we should expand about three variables?

4. **Multiplication and Squaring**

In this problem we will study the relationship between multiplication and squaring of integers.

Suppose that you are given the compute element $SQUARE$ that for an input X (an integer) outputs X^2. Also, you are given the compute elements ADD and SUB. ADD gets as inputs the integers X and Y and outputs $X+Y$. SUB gets as inputs the integers X and Y and outputs $X-Y$.

(a) Let $Z = X^2$, for an *arbitrary* input $X = (x_{n-1}, x_{n-2}, \ldots, x_1, x_0)$ with $Z = (z_{m-1}, z_{m-2}, \ldots, z_1, z_0)$, such that

$$Z = \sum_{i=0}^{m-1} z_i 2^i.$$

Prove that $z_0 = x_0$ and that $z_1 = 0$.

(b) Show how to compute the product of two integers using the circuit that consists of the compute elements ADD, SUB and $SQUARE$.

5. **Computing the $MAXIMUM$ function**

   In this problem we learn how to implement the $MAXIMUM$ function using LT circuits with arbitrary size weights.

   Let $Z_1, Z_2, \cdots, Z_n$ be n nonnegative integers with m bits each. Namely, $Z_i = (z_{i,m-1}, z_{i,m-2}, \ldots, z_{i,0})$ and

   $$Z_i = \sum_{j=0}^{m-1} z_{i,j} 2^j$$

   for $1 \leq i \leq n$.

   The $MAXIMUM$ function gets as inputs the n integers, Z_i, $1 \leq i \leq n$, and outputs the integer $\max_{1 \leq i \leq n}(Z_i)$.

   (a) We define the function $FIRST$ as follows:

   $$FIRST(Z_1, Z_2, \ldots, Z_n) = \begin{cases} 1 & MAXIMUM(Z_1, Z_2, \ldots, Z_n) = Z_1 \\ 0 & \text{otherwise} \end{cases}$$

   Prove that $FIRST \in LT_2$. What is the size of the circuit? What is the size of the circuit for $n = 2$ and arbitrary m?

   (b) Describe a construction of an LT_4 circuit for the $MAXIMUM$ function. What is the size of the circuit?

   (c) Draw the circuit for the case $n = 3$ and $m = 4$.

   (d) Is $MAXIMUM \in LT_3$?