Homework # 2

Due Tuesday, January 30, 2001, at 2:00 PM

1. **Unate Functions**

   In this problem we define the notion of unateness and study its relationship with linear threshold functions.

   DEFINITION. A function $f(x_1, x_2, ..., x_n)$ is *positive* in x_i iff it is possible to write a sum-of-products expression for f in which $\bar{x}_i$ does not appear.

   DEFINITION. A function $f(x_1, x_2, ..., x_n)$ is *negative* in x_i iff it is possible to write a sum-of-products expression for f in which x_i does not appear.

   DEFINITION. A function $f(x_1, x_2, ..., x_n)$ is a *unate function* iff it is positive or negative in each of its variables.

   For example, the function $x_1x_2 + x_1\bar{x}_3 + \bar{x}_3x_4$ is unate.

   (a) Prove that all unate functions of 2 or less variables are linear threshold functions.

   (b) Prove that all linear threshold functions, for an arbitrary number of variables, are unate functions.

   (c) Find a 4-variable unate function that is not a linear threshold function. Note that you also need to prove that the function you found is not a linear threshold function.

   (d) Prove or disprove: all unate functions of 3 variables are also linear threshold functions.

2. **Efficient Combining of Symmetric Functions**

   In this problem we learn how to implement linear threshold circuits that combine symmetric functions.

   (a) Let

   $$f_1(X) = XOR(x_1, x_2, x_3, x_4, x_5) = \begin{cases} 1 & \text{if } |X| \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

   Draw the depth-2 *layered* LT circuit with 6 elements that computes $f_1(X)$.

(b) Let

$$f_2(x_1, x_2, x_3, x_4, x_5) = \begin{cases} 1 & \text{if } |X| \in \{0, 1, 5\} \\ 0 & \text{otherwise} \end{cases}$$

Draw the depth-2 *layered* LT circuit with 3 elements that computes $f_2(X)$.

(c) Let

$$f_3(x_1, x_2, \cdots, x_{10}) = AND(f_1(x_1, x_2, \cdots, x_5), f_2(x_6, x_7 \cdots, x_{10})).$$

Use the constructions in (a) and (b) above to describe a depth-2 *layered* LT circuit with 8 elements that computes $f_3(X)$, $X \in \{0, 1\}^{10}$.

(d) Prove the general statement associated with (c) above. Namely, if $f_i(X)$, $1 \leq i \leq k$, are k symmetric functions that can be implemented with a *layered* LT circuit of size m_i, $1 \leq i \leq k$, respectively. Then for every linear threshold function $l(x_1, x_2, \cdots, x_k)$, the function $l(f_1(X), f_2(X), \cdots, f_k(X))$ can be implemented using a depth-2 LT circuit with $(1 - k + \sum_{i=1}^{k} m_i)$ elements.

3. **The Complete Quadratic Function**

The Complete Quadratic (CQ) function is the Boolean function that consists of the XOR of all the $\binom{n}{2}$ possible AND's between pairs of variables. Namely,

$$CQ(X) = (x_1 \wedge x_2) \oplus (x_1 \wedge x_3) \oplus \cdots (x_{n-1} \wedge x_n).$$

For example,

$$CQ(x_1, x_2, x_3) = (x_1 \wedge x_2) \oplus (x_1 \wedge x_3) \oplus (x_2 \wedge x_3).$$

(a) Prove that $CQ(X)$ is a symmetric function. Express $CQ(X)$ as a function of $|X|$.

(b) For which n (number of inputs), $CQ(X) \notin LT_1$? Prove your claim.

(c) Let $|f(X)|$ be the number of X's for which $f(X) = 1$. Calculate $|CQ(X)|$ for $X \in \{0, 1\}^n$.

(d) Draw the depth-2 *layered* LT circuit that computes $CQ(X)$ with $X \in \{0, 1\}^5$.

(e) Draw the depth-2 *non-layered* LT circuit that computes $CQ(X)$ with $X \in \{0, 1\}^5$.

(f) Prove that $CQ(x_1, x_2, \cdots, x_n)$ with $n = 2^k - 1$, can be implemented by an LT circuits with $k - 1$ gates. Draw the circuits for the cases $k = 2$ and $k = 3$.

4. **Generalizing the Non-layered Construction**

   A symmetric function $f(x_1, x_2, \cdots, x_n)$ can be described by a vector of length $n+1$ that we call $V(f) = (v_0, v_1, \cdots v_n)$ with

$$v_i = \begin{cases} 1 & \text{if for } |X| = i \ f(X) = 1 \\ 0 & \text{otherwise} \end{cases}$$

   Let $|V(f)|$ be the number of 1's in $V(f)$. In class we described a method for constructing a depth-2 *non-layered* LT circuit with $|V(f)|$ +1 elements. Now assume that you are given a symmetric function f with $V(f)$ consisting of k intervals of 1's. Generalize the non-layered approach and prove that f can be implemented by a depth-2 *non-layered* circuit with $k+1$ elements.

   - **Example:**
     Let $V(f) = \{011001010110\}$. Using the approach from class we can construct a depth-2 LT circuit with 7 elements ($|V(f)| = 6$). However, there are only 4 intervals of 1's. Namely, the general construction should provide a depth-2 circuit with 5 elements.
   - **Hint:**
     Look at the the paper "Linear-Input Logic", by R. C. Minnick, that was handed out in class.

5. **Size-Depth Trade-offs for the Implementation of Symmetric Functions**

   In this problem we will study the relationship between the size and the depth of LT circuits implementing symmetric functions.

   (a) In the case of *layered* circuits, is there any advantage in using depth greater than 2? (Assume that the elements of the circuit are in TH.)

   In the following sub-questions assume that the circuits are non-layered LT circuits.

   (b) Draw a 3-layer circuit that computes the XOR of 17 variables and has exactly 5 gates.

   (c) Generalize the above construction to a 3-layer circuit that implements the XOR of n variables, where $\frac{n+1}{2} = k^2$, k integer.

   (d) You are asked to design a 4-layer circuit that computes the XOR of n variables. Minimize the size of the circuit.

(e) Let $f(X)$ be a symmetric function with 29 variables with

$$V(f) = (011100101101110010110111001011).$$

Draw a circuit that implements $f(X)$ and has exactly 5 gates.

(f) Generalize the XOR construction in (c) and (d) to arbitrary depth d. What is the size of the circuit?

(g) Suppose one wants to implement a given symmetric function with the least possible size and arbitrary depth. Following the idea in (e), suggest a procedure one should follow.