Homework # 1

Due Tuesday, January 16, 2001, at 2:00 PM

1. **Basic Properties of Boolean Functions**
   In this problem we prove the basic properties associated with the Boolean functions AND, OR, NOT and XOR. We use the following notation: $AND(x_1, x_2) = x_1 x_2$, $OR(x_1, x_2) = x_1 + x_2$, $NOT(x_1) = \bar{x}_1$ and $XOR(x_1, x_2) = x_1 \oplus x_2$. Prove the following properties:

   - **Associativity**

   $$x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3$$
   $$x_1(x_2 x_3) = (x_1 x_2) x_3$$
   $$x_1 \oplus (x_2 \oplus x_3) = (x_1 \oplus x_2) \oplus x_3$$

   - **Commutativity**

   $$x_1 + x_2 = x_2 + x_1$$
   $$x_1 x_2 = x_2 x_1$$
   $$x_1 \oplus x_2 = x_2 \oplus x_1$$

   - **Distributivity**

   $$x_1(x_2 + x_3) = (x_1 x_2) + (x_1 x_3)$$
   $$x_1(x_2 \oplus x_3) = (x_1 x_2) \oplus (x_1 x_3)$$
   $$x_1 + (x_2 x_3) = (x_1 + x_2)(x_1 + x_3)$$

   - **DeMorgan's Rules**

   $$\overline{(x_1 + x_2)} = \bar{x}_1 \bar{x}_2$$
   $$\overline{(x_1 x_2)} = \bar{x}_1 + \bar{x}_2$$

- **Absorption Rules**

$$x_1 + \bar{x}_1 = 1$$

$$x_1 + x_1 = x_1$$

$$x_1\bar{x}_1 = 0$$

$$x_1x_1 = x_1$$

$$x_1 + x_1x_2 = x_1(x_1 + x_2) = x_1$$

- **Substitution of Constants**

$$x_1 + 0 = x_1$$

$$x_1 + 1 = 1$$

$$x_1 0 = 0$$

$$x_1 1 = x_1$$

$$x_1 \oplus 0 = x_1$$

$$x_1 \oplus 1 = \bar{x}_1$$

2. **Arbitrary Boolean Symbols**
In this problem we study the representation of the Boolean variables by values which are not 0 and 1.

(a) Assume that the Boolean variables are represented by 1's and -1's via the mapping $0 \to 1$ and $1 \to -1$. For example,

$$AND(x_1, x_2, \cdots, x_n) = \begin{cases} 1 & \text{if the number of -1's in } X \text{ is } n \\ 0 & \text{otherwise} \end{cases}$$

and

$$OR(x_1, x_2, \cdots, x_n) = \begin{cases} 0 & \text{if the number of 1's in } X \text{ is } n \\ 1 & \text{otherwise} \end{cases}$$

Derive the linear threshold representation of $AND(x_1, x_2, \cdots, x_n)$ and $OR(x_1, x_2, \cdots, x_n)$, for $X \in \{1, -1\}^n$.

(b) Use (a) above to derive the linear threshold representation of an arbitrary minterm, for $X \in \{1, -1\}^n$. What is the linear threshold representation of $AND(\bar{x}_1, x_2, \bar{x}_3, x_4, x_5)$ (for $X \in \{1, -1\}^n$)?

(c) Let

$$f(X) = sgn(w_0 + \sum_{i=1}^{n} w_i x_i)$$

be the linear threshold representation of a Boolean function $f(X)$ using the variables 0 and 1. We want to represent the Boolean variables by a's and b's via the mapping $0 \to a$ and $1 \to b$. Derive the linear threshold representation of $f(X)$ for $X \in \{a, b\}^n$.

(d) Let $X \in \{a, b\}^n$ and let

$$XOR(x_1, x_2, \cdots, x_n) = \begin{cases} 1 & \text{if the number of b's in } X \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

Prove that, for arbitrary a and b, $a \neq b$, there is no linear threshold representation for XOR.
Hint: You can use the fact that XOR is not a linear threshold function for $a = 0$ and $b = 1$.

3. **Simple Threshold Functions**
In this problem we derive the implementation of the symmetric Boolean functions that are in LT_1 (the class of linear threshold functions). Assume that $X \in \{0, 1\}^n$, and $|X|$ denotes the number of 1's in X. Let $0 \le i \le n$, and let

$$TH_i(x_1, x_2, \cdots, x_n) = \begin{cases} 1 & \text{if } |X| \ge i \\ 0 & \text{otherwise} \end{cases}$$

and

$$\overline{TH}_i(x_1, x_2, \cdots, x_n) = \begin{cases} 1 & \text{if } TH_i(X) = 0 \\ 0 & \text{otherwise} \end{cases}$$

We define the class of Boolean functions $TH = \{TH_i, \overline{TH}_i \mid 0 \le i \le n\}$.

(a) Prove that $AND(X) \in TH$, $OR(X) \in TH$ and $XOR(X) \notin TH$.

(b) Prove that $TH \subseteq LT_1$.

4. **Multilevel Threshold Functions**
In this problem we explore functions with nonbinary inputs. Assume that the inputs $x_i \in \{0, \ldots, q-1\}$ (the binary case is when $q = 2$).

We define

$$OR_q(x_1, x_2, \ldots, x_n) = \begin{cases} 0 & \text{if } x_i = 0 \text{ for all } 1 \le i \le n \\ 1 & \text{otherwise} \end{cases}$$

and

$$AND_q(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{if } x_i \neq 0 \text{ for all } 1 \leq i \leq n \\ 0 & \text{otherwise} \end{cases}$$

(a) For which $q \geq 2$ $OR_q \in LT_1$?

(b) For which $q \geq 2$ $AND_q \in LT_1$?