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Homework Assignment 1, Solutions

Problem 1.

(a) Maximum-likelihood decision rule: Decide that (+1,+1,…,+1) is transmitted if and only if: $\Pr((y_1,y_2,\ldots,y_n)\text{ is received}|(x_1,x_2,\ldots,x_n)=(+1,+1,\ldots,+1))$
$\geq \Pr((y_1,y_2,\ldots,y_n)\text{ is received}|(x_1,x_2,\ldots,x_n)=(-1,-1,\ldots,-1))$.

Since $(y_1,y_2,\ldots,y_n)$ are i.i.d. Gaussians with mean x_i and variance σ^2, the decision rule is equivalent to: Decide that (+1,+1,…,+1) is transmitted if and only if:

$$\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(y_i-1)^2}{2\sigma^2}} \geq \prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(y_i+1)^2}{2\sigma^2}} \Leftrightarrow e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-1)^2} \geq e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i+1)^2}$$

$$\Leftrightarrow \sum_{i=1}^{n}(y_i-1)^2 \leq \sum_{i=1}^{n}(y_i+1)^2 \Leftrightarrow \sum_{i=1}^{n}y_i \geq 0.$$

So the maximum-likelihood decoding algorithm is:

Decide that (+1,+1,…,+1) is transmitted if and only if $\sum_{i=1}^{n}y_i \geq 0$.

(b) Define:

$$p_{(+1)} \triangleq \Pr\{(x_1,x_2,\ldots,x_n)=(+1,+1,\ldots,+1)\},\quad p_{(-1)} \triangleq \Pr\{(x_1,x_2,\ldots,x_n)=(-1,-1,\ldots,-1)\}.$$

Then clearly $p_{(+1)}+p_{(-1)}=1$, and the decoder error probability is:

$$P_{error}=\Pr\{\sum_{i=1}^{n}y_i<0\,|\,(x_1,x_2,\ldots,x_n)=(+1,+1,\ldots,+1)\}\cdot p_{(+1)}+$$

$$\Pr\{\sum_{i=1}^{n}y_i\geq 0\,|\,(x_1,x_2,\ldots,x_n)=(-1,-1,\ldots,-1)\}\cdot p_{(-1)}.$$

We know that $\sum_{i=1}^{n}y_i$ is Gaussian with mean +n or –n and variance $n\sigma^2$, so

$$\Pr\{\sum_{i=1}^{n}y_i<0\,|\,(x_1,x_2,\ldots,x_n)=(+1,+1,\ldots,+1)\}$$

$$=\Pr\{\frac{\sum_{i=1}^{n}y_i-n}{\sqrt{n}\sigma}<-\frac{n}{\sqrt{n}\sigma}\,|\,(x_1,x_2,\ldots,x_n)=(+1,+1,\ldots,+1)\}=Q(\frac{\sqrt{n}}{\sigma})=Q(\sqrt{\frac{2E_b}{N_0}}).$$

Similarly, $\Pr\{\sum_{i=1}^{n}y_i\geq 0\,|\,(x_1,x_2,\ldots,x_n)=(-1,-1,\ldots,-1)\}=Q(\sqrt{\frac{2E_b}{N_0}})$.

Therefore $P_{error}=Q(\sqrt{\frac{2E_b}{N_0}})\cdot(p_{(+1)}+p_{(-1)})=Q(\sqrt{\frac{2E_b}{N_0}})$.

(c) There error probability of uncoded BPSK is also $Q(\sqrt{\frac{2E_b}{N_0}})$. So they have the same performance.

Problem 2.

Let Π be the permutation matrix of the interleaver [(*)], then the generator matrix of the (8,4) code is $(G, \Pi G)$, which is a 4 by 8 matrix. Notice that each row of G has weight 2, and each row of ΠG is also a row in G, so each row of $(G, \Pi G)$ —which is a codeword—has weight 4. So we know the minimum distance of the (8,4) code is at most 4.

Let $\Pi = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, then the generator matrix becomes:

$(G, \Pi G) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$, which has rank 4. So the code with Π as the interleaver will have dimension 4. And by checking the codewords we find that the minimum weight of all the n on-zero codewords is 4. So $\Pi = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ is a "best" interleaver we are looking for.

([(*)] Note: ‘ *Π is the permutation matrix of the interleaver’* means that if the input of the interleaver is (u_1, u_2, u_3, u_4), then the output of the interleaver is $(u_1, u_2, u_3, u_4)\Pi$.)

(Note: A more careful analysis of the code will show that there exist totally four "best" interleavers, whose corresponding permutation matrices are:

$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.)

Problem 3.

$$\Pr\{u_i = a \mid Y = y\} = \frac{1}{\Pr\{Y = y\}} \sum_{u:u_i=a} \Pr\{Y = y \mid U = u\} \cdot \Pr\{U = u\}$$

$$= \frac{1}{\Pr\{Y = y\}} p^0(a) \sum_{u:u_i=a} \Pr\{Y = y \mid X = uG\} \prod_{j \neq i} p^0(u_j).$$

Let $Q_i(a) \underline{\underline{\Delta}} \sum_{u:u_i=a} \Pr\{Y = y \mid X = uG\} \prod_{j \neq i} p^0(u_j),$ then

$$Q_1(a) = \Pr\{Y = y \mid X = (a,0)G\} p^0(0) + \Pr\{Y = y \mid X = (a,1)G\} p^0(1),$$

$$Q_2(a) = \Pr\{Y = y \mid X = (0,a)G\} p^0(0) + \Pr\{Y = y \mid X = (1,a)G\} p^0(1).$$

Then the *a posteriori* probability for u_i is

$$\Lambda_i = \log \frac{\Pr\{u_i = 0 \mid Y = y\}}{\Pr\{u_i = 1 \mid Y = y\}} = \log \frac{p^0(0)}{p^0(1)} + \log \frac{Q_i(0)}{Q_i(1)},$$

and the "extrinsic information" for u_i is $\Lambda_i^{(ext)} = \log \frac{Q_i(0)}{Q_i(1)}.$

(1) $p^0(0) = p^0(1) = \frac{1}{2}, y = ABCD.$ We have $Q_1(0) = \frac{5}{2^{10}}$, $Q_1(1) = \frac{17}{2^{13}}$, $Q_2(0) = \frac{3}{2^{10}}$, $Q_2(1) = \frac{33}{2^{13}}$. So $\Lambda_1 = \log \frac{40}{17}$, $\Lambda_2 = \log \frac{8}{11}$, $\Lambda_1^{(ext)} = \log \frac{40}{17}$, $\Lambda_2^{(ext)} = \log \frac{8}{11}$.

(2) $p^0(0) = \frac{1}{3}, p^0(1) = \frac{2}{3}, y = ABCD.$ We have $Q_1(0) = \frac{3}{2^9}$, $Q_1(1) = \frac{3}{2^{11}}$, $Q_2(0) = \frac{5}{3 \times 2^9}$, $Q_2(1) = \frac{17}{3 \times 2^{11}}$. So $\Lambda_1 = \log 2$, $\Lambda_2 = \log \frac{10}{17}$, $\Lambda_1^{(ext)} = \log 4$, $\Lambda_2^{(ext)} = \log \frac{20}{17}$.

Problem 4.

(a) Proof: Suppose there are m paths from u to x —P_1, P_2, …, P_m, and there are n paths from y to v—Q_1, Q_2, …, Q_n. Then the set of paths from u to v through edge e is:

$$\{P_i e Q_j \mid i = 1,2,...,m.\ j = 1,2,...,n.\}.$$

$$\therefore \mu_e(u,v) = \sum_{i=1}^{m} \sum_{j=1}^{n} w(P_i e Q_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} w(P_i) w(e) w(Q_j) \qquad \cdots\cdots(1)$$

$$= \left(\sum_{i=1}^{m} w(P_i) \right) w(e) \left(\sum_{j=1}^{n} w(Q_j) \right) \qquad \cdots\cdots(2)$$

$$= \mu(u,x) w(e) \mu(y,v).$$

Q.E.D.

(b) If we use formula (1), we'll have 2mn multiplications and mn-1 additions.
If we use formula (2), we'll have only 2 multiplications and m+n-2 additions.