EE/Ma 127c Error-Correcting Codes - Homework Assignment 2

Ling Li, ling@cs.caltech.edu

April 19, 2001

2.1 The trellis graph shows

$$W_1 = \begin{bmatrix} 1 & 2 \end{bmatrix}, W_2 = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 8 & 0 \end{bmatrix}, W_3 = \begin{bmatrix} 4 & 0 & 2 \\ 8 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}, W_4 = \begin{bmatrix} 2 & 0 \\ 4 & 2 \\ 0 & 1 \end{bmatrix}, W_5 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(a) From $\alpha_0 = 1$, $\alpha_i = \alpha_{i-1} W_i$, for $i = 1, \ldots, 5$, we get

$$\alpha_0 = 1, \alpha_1 = \begin{bmatrix} 1 & 2 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 8 & 16 & 2 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 160 & 18 & 56 \end{bmatrix}, \alpha_4 = \begin{bmatrix} 392 & 92 \end{bmatrix}, \alpha_5 = 576;$$

from $\beta_5 = 1$, $\beta_{i-1} = W_i \beta_i$, for $i = 1, \ldots, 5$, we get

$$\beta_0 = 576, \beta_1 = \begin{bmatrix} 80 \\ 248 \end{bmatrix}, \beta_2 = \begin{bmatrix} 12 \\ 28 \\ 16 \end{bmatrix}, \beta_3 = \begin{bmatrix} 2 \\ 8 \\ 2 \end{bmatrix}, \beta_4 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \beta_5 = 1.$$

(b) For $i = 0, 1, \ldots, 5$, and $v \in V_i$, let μ_i be a column vector of dimension q_i (same as β_i), and $\mu_i(v)$ be the flow from A to B through v. Then $\mu_i(v) = \mu_v(A, B) = \alpha_i(v)\beta_i(v)$.

$$\mu_0 = 576, \mu_1 = \begin{bmatrix} 80 \\ 496 \end{bmatrix}, \mu_2 = \begin{bmatrix} 96 \\ 448 \\ 32 \end{bmatrix}, \mu_3 = \begin{bmatrix} 320 \\ 144 \\ 112 \end{bmatrix}, \mu_4 = \begin{bmatrix} 392 \\ 184 \end{bmatrix}, \mu_5 = 576.$$

For $i = 1, \ldots, 5$, and edge $e = (u, v)$ where $u \in V_{i-1}$ and $v \in V_i$, we have $\mu_e(A, B) = \alpha_{i-1}(u) w(e) \beta_i(v)$. To avoid plotting the trellis graph, let ν_i be a $q_{i-1} \times q_i$ matrix and $v_i(u, v)$ be the flow $\mu_e(A, B)$. Thus we have

$$\nu_1 = \begin{bmatrix} 80 & 496 \end{bmatrix}, \nu_2 = \begin{bmatrix} 48 & 0 & 32 \\ 48 & 448 & 0 \end{bmatrix}, \nu_3 = \begin{bmatrix} 64 & 0 & 32 \\ 256 & 128 & 64 \\ 0 & 16 & 16 \end{bmatrix}, \nu_4 = \begin{bmatrix} 320 & 0 \\ 72 & 72 \\ 0 & 112 \end{bmatrix}, \nu_5 = \begin{bmatrix} 392 \\ 184 \end{bmatrix}.$$

We can verify that $\mu_i(v)$ is the same the sum of the v^{th} column of ν_i.

(c) Take the log (base 2) of W_i, we get W_i':

$$W_1' = \begin{bmatrix} 0 & 1 \end{bmatrix}, W_2' = \begin{bmatrix} 2 & -\infty & 1 \\ 1 & 3 & -\infty \end{bmatrix}, W_3' = \begin{bmatrix} 2 & -\infty & 1 \\ 3 & 0 & 1 \\ -\infty & 0 & 2 \end{bmatrix}, W_4' = \begin{bmatrix} 1 & -\infty \\ 2 & 1 \\ -\infty & 0 \end{bmatrix}, W_5' = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Using the "log" forward-backward algorithm with max-sum, we get α_i' and β_i' below:

$$\alpha_0' = 0, \alpha_1' = \begin{bmatrix} 0 & 1 \end{bmatrix}, \alpha_2' = \begin{bmatrix} 2 & 4 & 1 \end{bmatrix}, \alpha_3' = \begin{bmatrix} 7 & 4 & 5 \end{bmatrix}, \alpha_4' = \begin{bmatrix} 8 & 5 \end{bmatrix}, \alpha_5' = 8;$$

$$\beta_0' = 8, \beta_1' = \begin{bmatrix} 5 \\ 7 \end{bmatrix}, \beta_2' = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}, \beta_3' = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \beta_4' = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \beta_5' = 0.$$

2.2 Let $g(\Delta)$ denote the approximation function of $f(\Delta)$, which takes the form $(\Delta_0 = 0, \Delta_4 = \infty)$

$$g(\Delta) = \begin{cases} y_1, & \Delta_0 \le \Delta < \Delta_1; \\ y_2, & \Delta_1 \le \Delta < \Delta_2; \\ y_3, & \Delta_2 \le \Delta < \Delta_3; \\ y_4, & \Delta_3 \le \Delta < \Delta_4. \end{cases}$$

The problem is: define an error measure between f and g approximation, denoted by $E(f,g)$, and find g that minimizes $E(f,g)$.

(a) Define the error measure as the maximum discrepancy between f and g:

$$E(f,g) = \sup_{\Delta \ge \Delta_0} |f(\Delta) - g(\Delta)|.$$

(We should use sup instead of max.) Since we know $f(\Delta)$ is a continuous and monotonously decreasing function of Δ, we get for $i = 1, \ldots, 4$,

$$\begin{aligned} \sup_{\Delta \in [\Delta_{i-1}, \Delta_i)} |f(\Delta) - g(\Delta)| &= \sup_{\Delta \in [\Delta_{i-1}, \Delta_i)} |f(\Delta) - y_i| \\ &= \max\left(|f(\Delta_{i-1}) - y_i|, |f(\Delta_i) - y_i|\right) \\ &\ge \frac{f(\Delta_{i-1}) - f(\Delta_i)}{2}, \end{aligned}$$

with equalitiy iff

$$y_i = \frac{f(\Delta_{i-1}) + f(\Delta_i)}{2}. \tag{1}$$

Thus

$$\begin{aligned} E(f,g) = \sup_{\Delta \ge \Delta_0} |f(\Delta) - g(\Delta)| &= \max_{i=1}^{4} \sup_{\Delta \in [\Delta_{i-1}, \Delta_i)} |f(\Delta) - g(\Delta)| \\ &\ge \frac{1}{4} \sum_{i=1}^{4} \sup_{\Delta \in [\Delta_{i-1}, \Delta_i)} |f(\Delta) - g(\Delta)| \\ &= \frac{1}{8} f(0) = \frac{\ln 2}{8}, \end{aligned}$$

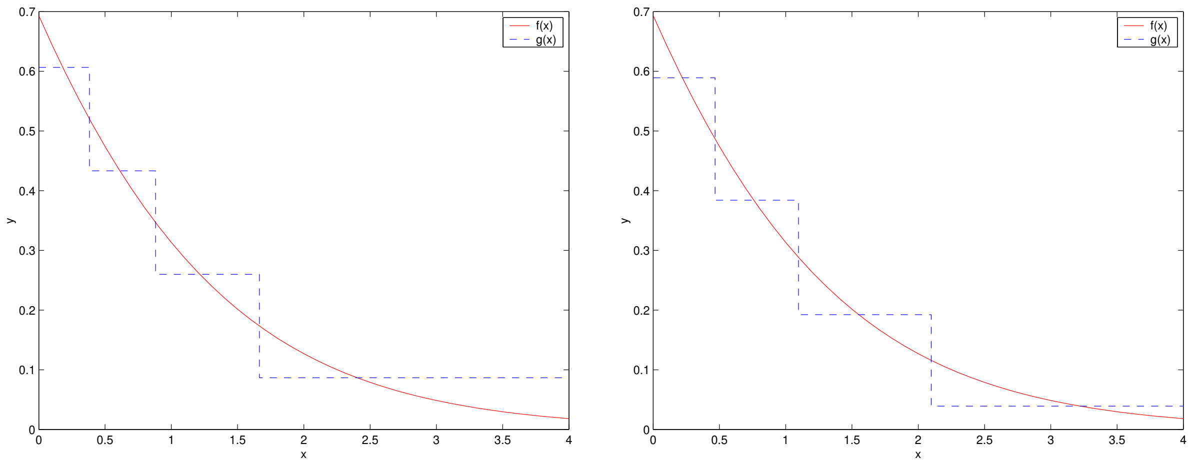
with equality iff (1) holds for all $i = 1, \ldots, 4$. To minimize $E(f,g)$, $f(\Delta_i)$ should be $\frac{4-i}{4} f(0)$, that is,

$$\Delta_i = f^{-1}\left(\frac{4-i}{4} \ln 2\right) = -\ln\left(2^{1-\frac{i}{4}} - 1\right),$$

and

$$y_i = \frac{9-2i}{8} f(0) = \frac{9-2i}{8} \ln 2.$$

Figure 1(a) shows both f and g.



(a) with maximum distance error

(b) with mean square error and $p = 1/5$.

Figure 1: $f(x) = \ln(1 + e^{-x})$ and its approximator $g(x)$.

(b) With some predefined probability $p(x)$, define the error measure as

$$E(f,g) = \int_0^\infty p(x)\left(f(x) - g(x)\right)^2 dx = \sum_{i=1}^{4} \int_{\Delta_{i-1}}^{\Delta_i} p(x)\left(f(x) - y_i\right)^2 dx.$$

To minimize $E(f,g)$, we have for $i = 1, \ldots, 4$ and $j = 1, \ldots, 3$,

$$\frac{\partial E}{\partial y_i} = 2\int_{\Delta_{i-1}}^{\Delta_i} p(x)\left(y_i - f(x)\right) dx = 0, \qquad (2)$$

$$\frac{\partial E}{\partial \Delta_j} = p(\Delta_j)\left(f(\Delta_j) - y_j\right)^2 - p(\Delta_j)\left(f(\Delta_j) - y_{j+1}\right)^2 = 0. \qquad (3)$$

From (2), we get

$$y_i = \frac{\int_{\Delta_{i-1}}^{\Delta_i} p(x) f(x) dx}{\int_{\Delta_{i-1}}^{\Delta_i} p(x) dx}.$$

It is reasonable to assume $p(\Delta_j) \neq 0$, and $y_j > f(\Delta_j) > y_{j+1}$ since $f(\Delta)$ is a decreasing function of Δ. Then from (3), we get

$$2f(\Delta_j) = y_j + y_{j+1}.$$

We can not get a closed form for y_i and Δ_i. However, we can use numerical techniques to get a numerical solution. For example, if

$$p(x) = \begin{cases} \frac{1}{5}, & 0 \le x \le 5; \\ 0, & \text{otherwise} \end{cases}$$

(since $f(x) < 0.007$ when $x > 5$, we might omit them) I got

$$y_1 \approx 0.589135, y_2 \approx 0.384069, y_3 \approx 0.192328, y_4 \approx 0.039135,$$
$$\Delta_1 \approx 0.467161, \Delta_2 \approx 1.096547, \Delta_3 \approx 2.098058.$$

Figure 1(b) shows that this g gives more attention to larger x than the previous one.

2.1 *Commutative semiring.*

(a) Below I list the equations of the distributive law in every case. It is really straightforward that they hold.

sum-product $(a \cdot b) + (a \cdot c) = a \cdot (b + c)$.

min-product $a > 0$. $\min(a \cdot b, a \cdot c) = a \cdot \min(b, c)$.

max-product $a > 0$. $\max(a \cdot b, a \cdot c) = a \cdot \max(b, c)$.

min-sum $\min(a + b, a + c) = a + \min(b, c)$.

max-sum $\max(a + b, a + c) = a + \max(b, c)$.

(b) From mP to MP, we can use the mapping $x \mapsto x^{-1}$. That is

mP	x	$(0, \infty]$	∞	1	$a \cdot b$	$\min(a, b)$
MP	x^{-1}	$[0, \infty)$	0	1	$a^{-1} \cdot b^{-1}$	$\max\left(a^{-1}, b^{-1}\right)$

From MP to mS, we can use the mapping $x \mapsto -\log x$.

MP	x	$[0, \infty)$	0	1	$a \cdot b$	$\max(a, b)$
mS	$-\log x$	$(-\infty, \infty]$	∞	0	$(-\log a) + (-\log b)$	$\min(-\log a, -\log b)$

From mS to MS, we can use the mapping $x \mapsto -x$.

mS	x	$(-\infty, \infty]$	∞	0	$a + b$	$\min(a, b)$
MS	$-x$	$[-\infty, \infty)$	$-\infty$	0	$(-a) + (-b)$	$\max(-a, -b)$

Then by inversing or composing mappings, we get the table below:

$$
\begin{array}{c c}
 & \begin{array}{cccc} \text{mP} & \text{MP} & \text{mS} & \text{MS} \end{array} \\
\begin{array}{c} \text{mP} \\ \text{MP} \\ \text{mS} \\ \text{MS} \end{array} &
\begin{pmatrix}
* & x^{-1} & \log x & -\log x \\
x^{-1} & * & -\log x & \log x \\
e^{x} & e^{-x} & * & -x \\
e^{-x} & e^{x} & -x & *
\end{pmatrix}
\end{array}
$$