EE/Ma 127a Error-Correcting Codes - Homework Assignment 3

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3.1 Gross-error

Here 'gross-error' means sometimes the memory will make disgusting errors that reset the code bits and check bits to all 1 or all 0, and if not with some bits inverted, all 0 is a valid codeword.

In order to detect such error, we can invert some check bits so that when the codeword is reset to 0, we can detect the error. For example, when `CB0` is inverted and a memory error makes the bits be all 0, the chip will read the code bits as 0 but `CB0` as 1. Thus we know some error happens. However, error syndrome in this situation is same as the one when `CB0` itself has an error. Inverting `CB1`, which produces syndrome '`LLHHHH`' when all bits are reset to 0, helps to distinguish those two situations.

However, for the similar reason, we should leave at least 2 check bits not inverted to detect and distinguish the gross-error of all 1.

3.2 Lower bounds

(a) $d = 5$. By Hamming bound, we have: $t = (d - 2)/2 = 2$,

$$2^{n-k} = \frac{2^n}{M} \geq V(n, t) = 1 + n + \binom{n}{2} = \frac{n^2 + n + 2}{2}. \tag{1}$$

That is

$$2^r \geq \frac{(k + r)^2 + k + r + 2}{2}. \tag{2}$$

(b) $d = 6$. If one bit is removed from code (n, k) with $d = 6$, then the minimum distance d' of that new code $(n - 1, k)$ is at least 5. Thus (1) is

$$2^{n-1-k} \geq \frac{(n - 1)^2 + (n - 1) + 2}{2},$$

i.e., for $d = 6$ we have

$$2^r \geq (k + r)^2 - (k + r) + 2. \tag{3}$$

(c) For $k = 32$, $d = 5$, using (2) we have $r \geq 10$; for $k = 32$, $d = 6$, using (3) we have $r \geq 11$. Figure 1 gives the curves for finding out the estimated lower bound of r.

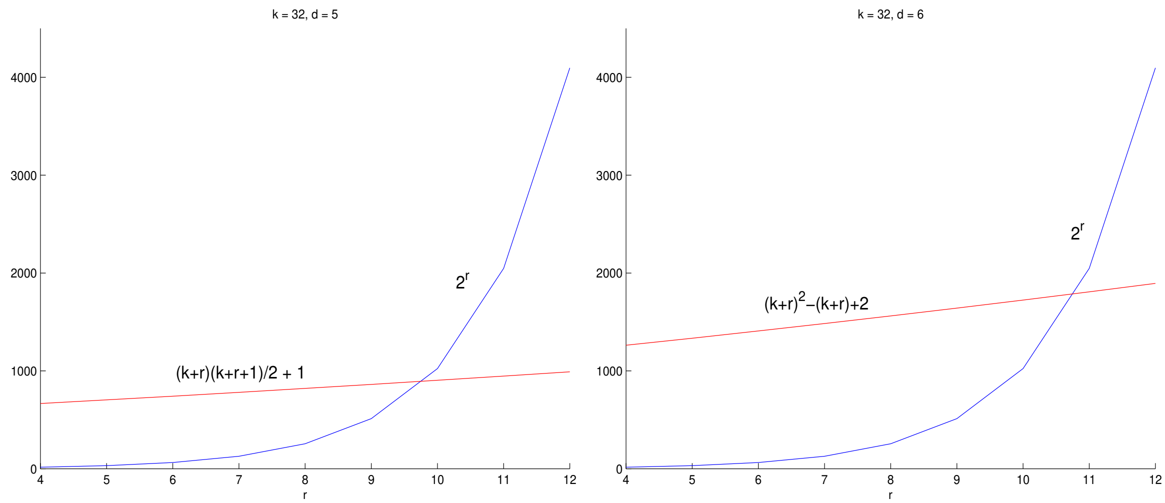

Figure 1: Curves for finding out estimated lower bound of r.

3.3 Weight enumerator

All the codewords of code $(2^m - 1, m)$ can be produced by xG, where x is any binary vector of dimension $1 \times m$, and G is the generator matrix for $(2^m - 1, m)$, which is also one parity-check matrix for the Hamming code $(2^m - 1, 2^m - 1 - m)$. So columns of G are distinct nonzero vectors of length m.

- For $x \neq 0$, there is at least one column vector g of G such that $xg = 1$. Then in the pairing of column vectors other than g,
$$g' \leftrightarrow g' + g,$$
if $xg' = 1$, then $x(g' + g) = 0$, and vice versa. So the weight of xG is
$$1 + \frac{(2^m - 1) - 1}{2} = 2^{m-1}.$$
- For $x = 0$, obviously the weight of xG is 0.

Since there're $(2^m - 1)$ nonzero x's and one zero x, the weight enumerator for code $(2^m - 1, m)$ is
$$A(z) = 1 + (2^m - 1) z^{2^{m-1}}. \tag{4}$$

3.4 MacWilliams identities

Let A_i be the number of words of weight i in the code $(2^m - 1, m)$, and B_j be the number of words of weight j in the code $(2^m - 1, 2^m - 1 - m)$, which is dual to $(2^m - 1, m)$. As shown in (4) of 3.3, we have $A_k = 0$ except $A_0 = 1$ and $A_{2^{m-1}} = 2^m - 1$. Using MacWilliams identities, we have
$$\begin{aligned} B_j &= \frac{1}{2^k} \sum_{i=0}^{n} A_i K_{i,j}^{(n)} \\ &= \frac{1}{2^m} \left[K_{0,j}^{(n)} + (2^m - 1) K_{2^{m-1},j}^{(n)} \right], \end{aligned}$$

where $n = 2^m - 1$, $k = m$, and $K_{i,j}^{(n)}$ is the coefficient of z^j in $(1-z)^i(1+z)^{n-i}$. Thus $K_{0,j}^{(n)} = \binom{n}{j}$, and since

$$\begin{aligned}(1-z)^{2^{m-1}}(1+z)^{n-2^{m-1}} &= (1-z)^{2^{m-1}}(1+z)^{2^{m-1}-1} \\ &= (1-z)(1-z^2)^{2^{m-1}-1},\end{aligned}$$

we also have

$$K_{2^{m-1},j}^{(n)} = \begin{cases} (-1)^s\binom{2^{m-1}-1}{s}, & \text{if } j = 2s \text{ is even;} \\ (-1)^s\binom{2^{m-1}-1}{s-1}, & \text{if } j = 2s-1 \text{ is odd.}\end{cases}$$

Thus

$$B_3 = \frac{1}{2^m}\left[\binom{n}{3} + (2^m-1)(-1)^2\binom{2^{m-1}-1}{1}\right] = \frac{(2^m-1)(2^{m-1}-1)}{3},$$

$$B_4 = \frac{1}{2^m}\left[\binom{n}{4} + (2^m-1)(-1)^2\binom{2^{m-1}-1}{2}\right] = \frac{(2^m-1)(2^{m-1}-1)(2^{m-2}-1)}{3}.$$

For example, $m = 3$, we have $B_3 = B_4 = 7$ for code $(7, 4)$.

3.5 Time and day of the week

We postulate that the time of day ranges $[0, 24)$, and the day of week ranges from 0 to 6, where 0 denotes Sunday, 1 denotes Monday, ...

Generally, after t hours (if $t < 0$, that is $(-t)$ hours ago) from noon (12 p.m.) on Wednesday, the time of day is

$$(12+t) \mod 24$$

and the day of week is

$$3 + \left\lfloor \frac{12+t}{24} \right\rfloor \mod 7.$$

(a) After a million hours. $t = 10^6$. The time of day is

$$12 + t = 4 \pmod{24}$$

and the day of week is

$$3 + \left\lfloor \frac{12+t}{24} \right\rfloor = 41670 = 6 \pmod 7.$$

That is, 4 a.m. in Saturday.

(b) A million hours ago. $t = -10^6$. The time of day is

$$12 + t = 20 \pmod{24}$$

and the day of week is

$$3 + \left\lfloor \frac{12+t}{24} \right\rfloor = -41664 = 0 \pmod 7.$$

That is, 8 p.m. in Sunday.