EE/Ma 126a Information Theory - Homework Set #4

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4.1 Bottleneck

(a) $X_1 \to X_2 \to X_3$, thus the data processing inequality says

$$\boxed{I(X_1;X_3) \le I(X_1;X_2) \le H(X_2) \le \log k}.$$

(b) $I(X_1;X_3) \le 0$ for $k = 1$. Since the information of two random variables should not be negative, $\boxed{I(X_1;X_3) = 0}$, which means X_1 and X_3 are independent.

4.2 Conditional entropy

The conditional mutual information is non-negative. Thus

$$H(X|g(Y)) - H(X|Y,g(Y)) = I(X;Y|g(Y)) \ge 0,$$

and

$$0 \le I(X;g(Y)|Y) \le H(g(Y)|Y) = 0.$$

So

$$H(X|Y) - H(X|Y,g(Y)) = I(X;g(Y)|Y) = 0.$$

Thus we get $H(X|g(Y)) \ge H(X|Y)$ with equality iff $I(X;Y|g(Y)) = 0$, i.e., $H(X|g(Y)) = H(X|Y)$ iff $\boxed{X \text{ and } Y \text{ are independent when given } g(Y)}$.

4.3 Random box size

$$\begin{aligned}
\ln \lim_{n\to\infty} V_n^{1/n} &= \lim_{n\to\infty} \ln V_n^{1/n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln X_i \\
&= E \ln X_1 \text{ with probability } 1 \\
&= \int_0^1 \ln x \, dx = x\,(\ln x - 1)\big|_0^1 = -1.
\end{aligned}$$

Thus

$$\boxed{\lim_{n\to\infty} V_n^{1/n} = \mathrm{e}^{-1} \text{ with probability } 1}$$

while

$$\boxed{(EV_n)^{1/n} = \left(\prod_{i=0}^{n} EX_i\right)^{\frac{1}{n}} = EX_1 = \frac{1}{2}}.$$

4.4 Uniquely decodable and instantaneous codes

Any instantaneous code is also a uniquely decodable code, thus

$$L_1 \geq L_2.$$

From Kraft inequality and McMillan inequality we know that for any uniquely decodable code, there exists an instantaneous code with the same code-lengths. So we have

$$L_1 \leq L_2.$$

Thus $\boxed{L_1 = L_2}$.

4.5 Code types

Characteristics of those codes and the reasons are listed below.

Code	Instantaneous	Uniquely decodable	Non-singular
(a) $\{0, 10\}$	$\surd$	$\surd$	$\surd$
(b) $\{0, 10, 11\}$	$\surd$	$\surd$	$\surd$
(c) $\{0, 01, 11\}$		$\surd$	$\surd$
(d) $\{0, 01, 10\}$			$\surd$
(e) $\{0, 01\}$		$\surd$	$\surd$
(f) $\{00, 01, 10, 11\}$	$\surd$	$\surd$	$\surd$
(g) $\{110, 11, 10\}$		$\surd$	$\surd$
(h) $\{110, 11, 100, 00, 10\}$		$\surd$	$\surd$

(a),(b),(f): This is a prefix-free code. So it is instantaneous, uniquely decodable and non-singular.

(c),(e): This is a suffix-free code but not a prefix-free code. So it is non-instantaneous, uniquely decodable and non-singular.

(d): Ambiguous decoding. String 010 can be divided either as 0-10 or 01-0, thus not uniquely decodable and non-instantaneous. However, it is non-singular.

(g): The first number must be 1. If the second number is 0, the first codeword is 10. If the second one is 1, we cannot tell what the first codeword is at this point. We should look at the third number (if exists). If the third number is 0, then the codeword is 110, otherwise the codeword is 11. Thus this code is uniquely decodable but non-instantaneous.

(h): If the first number is 0, the codeword is 00. If the first number is 1, look at the second number. If the second number is 0, count the number of consecutive 0's after the first number. If the number is even, the codeword is 100, since the only codeword beginning with 0 is 00; otherwise 10. If the first two numbers are both 1, count the number of consecutive 0's after the second number. If the number is even, the codeword is 11 otherwise is 110. Thus this code is uniquely decodable but non-instantaneous.