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16.1 Cover time of K_n

By symmetry, the expected time to visit k (different) vertices doesn't depend on which k vertices are visited. Let T_k denote that expected time. Obviously $T_1 = 0$.

When k vertices have been visited, the walk could visit some new vertex with probability $\frac{n-k}{n-1}$, or visit some visited vertex (thus still stays within those k vertices) with probability $\frac{k-1}{n-1}$. Hence the expected time to visit $k+1$ vertices after having already visited k vertices is

$$T_{k+1} - T_k = 1 + \frac{k-1}{n-1}(T_{k+1} - T_k),$$

i.e.,

$$T_{k+1} - T_k = \frac{n-1}{n-k}.$$

Thus the expected time to cover the graph K_n is

$$T_n = \sum_{k=1}^{n-1}(T_{k+1} - T_k) + T_1 = (n-1)\sum_{k=1}^{n-1}\frac{1}{n-k} = (n-1)\sum_{k=1}^{n-1}\frac{1}{k}.$$

16.2 Lollipop graph

By symmetry, we can collapse all the vertices in the clique except $\frac{n}{2}$ into one super vertex S (Figure 1). Since S includes $\frac{n}{2}$ vertices, there is a self-loop for S with probability $\frac{\frac{n}{2}-1}{\frac{n}{2}} = \frac{n-2}{n}$, and

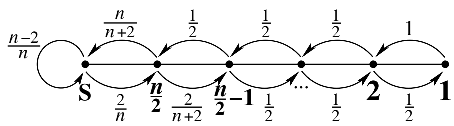

Figure 1: The n-vertex lollipop graph with collapsed super vertex S.

the probability from S to $\frac{n}{2}$ is $\frac{2}{n}$. Similarly, the probability from $\frac{n}{2}$ to S is $\frac{\frac{n}{2}}{\frac{n}{2}+1} = \frac{n}{n+2}$ and that from $\frac{n}{2}$ to $\frac{n}{2}-1$ is $\frac{2}{n+2}$. All these probabilities are also shown in the figure. Thus we can construct a series of equations on the hitting time $h_{i,1}$:

$$\begin{aligned}
h_{S,1} &= 1 + \frac{n-2}{n} h_{S,1} + \frac{2}{n} h_{\frac{n}{2},1}, &(1)\\
h_{\frac{n}{2},1} &= 1 + \frac{n}{n+2} h_{S,1} + \frac{2}{n+2} h_{\frac{n}{2}-1,1}, &(2)\\
h_{i,1} &= 1 + \frac{1}{2} h_{i+1,1} + \frac{1}{2} h_{i-1,1}, &(3)\\
h_{1,1} &= 0, &(4)
\end{aligned}$$

where $i = 2, \ldots, \frac{n}{2}-1$. Eliminating $h_{S,1}$ in (1) and (2), we get

$$h_{\frac{n}{2},1} - h_{\frac{n}{2}-1,1} = \frac{n^2+2n+4}{4}. \tag{5}$$

Write (3) as

$$h_{i,1} - h_{i-1,1} = h_{i+1,1} - h_{i,1} + 2. \tag{6}$$

Recursively using (6), we get

$$h_{i,1} - h_{i-1,1} = h_{\frac{n}{2},1} - h_{\frac{n}{2}-1,1} + 2\left(\frac{n}{2} - i\right) = \frac{n^2+6n+4}{4} - 2i.$$

Then, together with (4),

$$h_{\frac{n}{2},1} = \sum_{i=2}^{\frac{n}{2}} (h_{i,1} - h_{i-1,1}) + h_{1,1} = \left(\frac{n}{2} - 1\right)\left(\frac{n^2+6n+4}{4} - \frac{2+\frac{n}{2}}{2} \times 2\right) = \frac{(n-2)(n^2+4n-4)}{8}.$$

From the class notes, the hitting time $h_{1,\frac{n}{2}}$ is

$$h_{1,\frac{n}{2}} = \left(\frac{n}{2} - 1\right)^2 = \frac{(n-2)^2}{4},$$

which is different from $h_{\frac{n}{2},1}$. ($h_{\frac{n}{2},1} - h_{1,\frac{n}{2}} = \frac{n(n-2)(n+2)}{8}$, and the case $n = 2$ is too trivial.)

16.3 Random walk on a circle

(a) By symmetry, vertex i and $n-i$ have the same hitting times from 0, i.e., $h_{0,i} = h_{0,n-i}$, and the same probabilities to be visited last. So we can put C_n as in Figure 2. From now on we

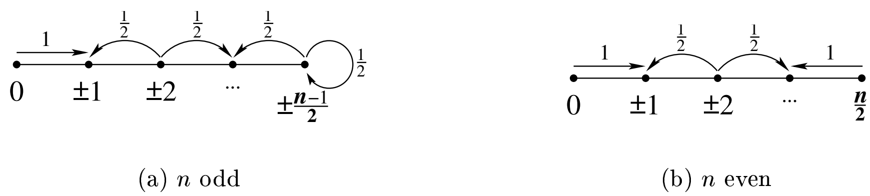

Figure 2: Make circle a line by symmetry.

regard vertex i and vertex $n-i$ as one vertex $\pm i$. For n even, $\pm\frac{n}{2}$ is just $\frac{n}{2}$.

From the figure, vertex $\pm(i+1)$ is always visited after vertex $\pm i$, if $\pm i$ is not at the right end of the line. So the last visited vertex must be $\frac{n}{2}$ when n is even, and $\pm\frac{n-1}{2}$ (or $\frac{n\pm 1}{2}$) when n is odd.

(b) By the same argue, we know that vertex $\frac{n}{2}$ when n is even, and $\frac{n\pm 1}{2}$ when n is odd, has the largest hitting time from 0.

To compute $h_{0,i}$, we can "hang" the circle by vertex i and get the Figure 3. For simplicity,

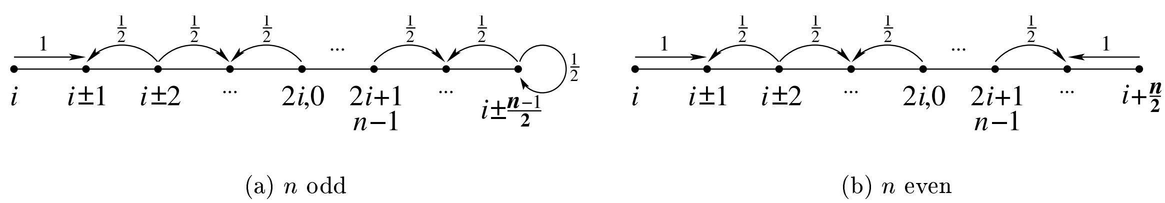

Figure 3: Hang the circle

we omit the $\bmod n$ in those subscripts.

- When n is odd, by conditioning on the first step the walk takes, we get the following relations:

$$\begin{aligned} h_{i\pm\frac{n-1}{2},i} &= 1 + \frac{1}{2}\left(h_{i\pm\frac{n-3}{2},i} + h_{i\pm\frac{n-1}{2},i}\right), \\ h_{i\pm j,i} &= 1 + \frac{1}{2}\left(h_{i\pm(j-1),i} + h_{i\pm(j+1),i}\right), \\ h_{i,i} &= 0, \end{aligned}$$

  where $1 \le j \le \frac{n-3}{2}$. Applying the same method as in problem 16.2, we can get $h_{i\pm j,i} = j(n-j)$ for $1 \le j \le \frac{n-1}{2}$. So, $h_{0,i} = i(n-i)$, with maximum value $\frac{n^2-1}{4}$ when $i = \frac{n\pm 1}{2}$. That is, vertex $\frac{n\pm 1}{2}$has the largest hitting time when n is odd.

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- When n is even, similarly, we can get the following relations:

$$\begin{aligned} h_{i+\frac{n}{2},i} &= 1 + h_{i\pm\left(\frac{n}{2}-1\right),i}, \\ h_{i\pm j,i} &= 1 + \frac{1}{2}\left(h_{i\pm(j-1),i} + h_{i\pm(j+1),i}\right), \\ h_{i,i} &= 0, \end{aligned}$$

where $1 \leq j \leq \frac{n}{2} - 1$. Also, using the same method, we have $h_{i\pm j,i} = j(n-j)$ for $1 \leq j \leq \frac{n}{2}$. So, $h_{0,i} = i(n-i)$, with maximum value $\frac{n^2}{4}$ when $i = \frac{n}{2}$. That is, vertex $\frac{n}{2}$ has the largest hitting time when n is odd.

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16.4 Walk from n to 1

Let $p(i|j)$ denote the conditional probability that i is visited given j is visited ($i \leq j$), and let $p(i)$ denote the probability that i is visited during the walk. It is obvious that $p(j|j) = 1$ and $p(i) = p(i|n)$ since n is the start vertex.

After visiting j, a vertex $k (< j)$ is visited with probability $\frac{1}{j-1}$. Then

$$p(i|j) = \frac{1}{j-1} \sum_{k=i}^{j-1} p(i|k). \tag{7}$$

For $i < n$ and $j = i + 1$, (7) gives

$$p(i|i+1) = \frac{1}{i} p(i|i) = \frac{1}{i}.$$

And for $i < j - 1$, we also have

$$p(i|j-1) = \frac{1}{j-2} \sum_{k=i}^{j-2} p(i|k),$$

thus (7) can be rearranged as

$$p(i|j) = \frac{1}{j-1} \sum_{k=i}^{j-2} p(i|k) + \frac{1}{j-1} p(i|j-1) = \frac{j-2}{j-1} p(i|j-1) + \frac{1}{j-1} p(i|j-1) = p(i|j-1).$$

Hence for $i \leq j - 1 < n$,

$$p(i|j) = p(i|j-1) = \ldots = p(i|i+1) = \frac{1}{i}. \tag{8}$$

(a) $p(n|n) = 1$, and from (8), $p(i|n) = \frac{1}{i}$ for $i < n$. Thus the probability that i is visited during the walk, i.e., $p(i) = p(i|n)$, is $\frac{1}{i}$ for $i = 1, 2, \ldots, n-1$ and 1 for $i = n$.

(b) Let random variable $X_i = 1$ if i is visited and $X_i = 0$ if i is not visited during the walk. Since any vertex can be visited at most once, the time to reach 1 is

$$T = \sum_{i=2}^{n} X_i.$$

From $E(X_i) = p(i)$, we get

$$E(T) = \sum_{i=2}^{n} E(X_i) = \sum_{i=2}^{n} p(i) = \sum_{i=2}^{n-1} \frac{1}{i} + 1 = \sum_{i=1}^{n-1} \frac{1}{i}. \tag{9}$$

(c) Without loss of generality, assume $i < j$. The probability that two different vertices i and j are both visited during the walk is $p(i|j)p(j)$, which is $\frac{1}{ij}$ if $j < n$ or $\frac{1}{i}$ if $j = n$.

(d) When there are self-loops (i, i), (7) becomes

$$p(i|j) = \frac{1}{j}\sum_{k=i}^{j} p(i|k) = \frac{1}{j}p(i|j) + \frac{1}{j}\sum_{k=i}^{j-1} p(i|k),$$

i.e.,

$$p(i|j) = \frac{1}{j-1}\sum_{k=i}^{j-1} p(i|k).$$

This is just the same as (7). Since we still have $p(i|i) = 1$ here, $p(i|j)$ remains the same as that in the graph without self-loops. Thus the answers to (a) and (c) do not change. However, since now one vertex could be visited several times, the answer to (b) changes.

Let T_i denote the number of visits to vertex i during a walk. Then the time to reach 1 is

$$T = \sum_{i=2}^{n} T_i.$$

Since there exist self-loops, the probability that walking from vertex i back to itself is $\frac{1}{i}$. Thus

$$E(T_i) = p(i) \cdot \left(1 + \frac{1}{i} + \frac{1}{i^2} + \cdots\right) = p(i) \cdot \frac{i}{i-1}.$$

Hence

$$E(T) = \sum_{i=2}^{n} E(T_i) = \sum_{i=2}^{n} p(i) \cdot \frac{i}{i-1} = \sum_{i=2}^{n-1} \frac{1}{i-1} + \frac{n}{n-1} = 1 + \sum_{i=1}^{n-1} \frac{1}{i}.$$

Compared to (9), the expected time to reach 1 is increased by 1.