CNS/CS/EE 188a Computation Theory and Neural Systems - Homework #1

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1.1 Basic Properties of Boolean Functions.

- **Associativity**. Let $OR_1 = x_1 + (x_2 + x_3)$, $OR_2 = (x_1 + x_2) + x_3$, $AND_1 = x_1(x_2x_3)$, $AND_2 = (x_1x_2)x_3$, $XOR_1 = x_1 \oplus (x_2 \oplus x_3)$, and $XOR_2 = (x_1 \oplus x_2) \oplus x_3$. From the table

x_1	x_2	x_3	OR_1	OR_2	AND_1	AND_2	XOR_1	XOR_2
0	0	0	0	0	0	0	0	0
0	0	1	1	1	0	0	1	1
0	1	0	1	1	0	0	1	1
0	1	1	1	1	0	0	0	0
1	0	0	1	1	0	0	1	1
1	0	1	1	1	0	0	0	0
1	1	0	1	1	0	0	0	0
1	1	1	1	1	1	1	1	1

  we get $OR_1 = OR_2$, $AND_1 = AND_2$, and $XOR_1 = XOR_2$. That is, $x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3$, $x_1(x_2x_3) = (x_1x_2)x_3$, and $x_1 \oplus (x_2 \oplus x_3) = (x_1 \oplus x_2) \oplus x_3$.

- **Commutativity**. By definitions, AND, OR, and XOR are functions only of $|X|$. Thus they are symmetric and the order of the inputs does not matter. So

$$x_1 + x_2 = x_2 + x_1, \quad x_1x_2 = x_2x_1, \quad x_1 \oplus x_2 = x_2 \oplus x_1.$$

- **Distributivity**. From rules in "Commutativity" and "Substitution of Constants," we have

$$\begin{aligned} 0(x_2 + x_3) &= (x_2 + x_3)0 = 0 = 0 + 0 = (x_20) + (x_30) = (0x_2) + (0x_3), \\ 1(x_2 + x_3) &= (x_2 + x_3)1 = x_2 + x_3 = (x_21) + (x_31) = (1x_2) + (1x_3). \end{aligned}$$

  Thus $x_1(x_2 + x_3) = (x_1x_2) + (x_1x_3)$. Similarly, we have

$$\begin{aligned} &0(x_2 \oplus x_3) = 0 = 0 \oplus 0 = (0x_2) \oplus (0x_3), \quad && 1(x_2 \oplus x_3) = x_2 \oplus x_3 = (1x_2) \oplus (1x_3), \\ &0 + (x_2x_3) = x_2x_3 = (0 + x_2)(0 + x_3), && 1 + (x_2x_3) = 1 = 1 \cdot 1 = (1 + x_2)(1 + x_3). \end{aligned}$$

  Thus $x_1(x_2 \oplus x_3) = (x_1x_2) \oplus (x_1x_3)$ and $x_1 + (x_2x_3) = (x_1 + x_2)(x_1 + x_3)$.

- **DeMorgan's Rules**.

x_1	x_2	$\overline{(x_1 + x_2)}$	$\bar{x}_1\bar{x}_2$	$\overline{(x_1x_2)}$	$\bar{x}_1 + \bar{x}_2$	$x_1 + x_1x_2$	$x_1(x_1 + x_2)$
0	0	1	1	1	1	0	0
0	1	0	0	1	1	0	0
1	0	0	0	1	1	1	1
1	1	0	0	0	0	1	1

  Thus $\overline{(x_1 + x_2)} = \bar{x}_1\bar{x}_2$ and $\overline{(x_1x_2)} = \bar{x}_1 + \bar{x}_2$.

- **Absorption Rules**.

x_1	$x_1 + x_1$	$x_1 x_1$	$\bar{x}_1$	$x_1 + \bar{x}_1$	$x_1 \bar{x}_1$
0	0	0	1	1	0
1	1	1	0	1	0

So $x_1 + \bar{x}_1 = 1$, $x_1 + x_1 = x_1$, $x_1\bar{x}_1 = 0$, and $x_1 x_1 = x_1$. And from the table in "DeMorgan's Rules," we get $x_1 + x_1 x_2 = x_1(x_1 + x_2) = x_1$.

- **Substitution of Constants**.

x_1	$x_1 + 0$	$x_1 1$	$x_1 \oplus 0$	$x_1 + 1$	$x_1 0$	$x_1 \oplus 1$	$\bar{x}_1$
0	0	0	0	1	0	1	1
1	1	1	1	1	0	0	0

Thus

$$x_1 + 0 = x_1 1 = x_1 \oplus 0 = x_1, \quad x_1 + 1 = 1, \quad x_1 0 = 0, \quad x_1 \oplus 1 = \bar{x}_1.$$

1.2 Arbitrary Boolean Symbols.

(a) The number of -1's in X (which is the number of 1's in X if values $\{0, 1\}$ is used) is

$$|X| = \sum_{i=1}^{n} \frac{1 - x_i}{2} = \frac{1}{2}(n - \sum_{i=1}^{n} x_i).$$

Thus the linear threshold representations of AND and OR are[1]

$$AND(x_1, x_2, \dots, x_n) = \operatorname{sgn}(|X| - n) = \operatorname{sgn}(2|X| - 2n) = \operatorname{sgn}(-n - \sum_{i=1}^{n} x_i), \tag{1}$$

$$OR(x_1, x_2, \dots, x_n) = \operatorname{sgn}(|X| - 1) = \operatorname{sgn}(2|X| - 2) = \operatorname{sgn}(n - 2 - \sum_{i=1}^{n} x_i).$$

(b) For $\text{minterm}(x_1, x_2, \dots, x_n | c_1, c_2, \dots, c_n) = x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n}$, its value is 1 iff $x_i = c_i$ for all $i = 1, 2, \dots, n$. Since $x_i \in \{-1, 1\}$ and $c_i \in \{-1, 1\}$, we have

$$-c_i x_i = \begin{cases} -1, & \text{if } x_i = c_i; \\ 1, & \text{otherwise.} \end{cases}$$

Thus by the linear threshold representation of AND in (1),

$$\begin{aligned} \text{minterm}(x_1, x_2, \dots, x_n | c_1, c_2, \dots, c_n) &= AND(-c_1 x_1, -c_2 x_2, \dots, -c_n x_n) \\ &= \operatorname{sgn}(\sum_{i=1}^{n} c_i x_i - n). \end{aligned}$$

For example,

$$\begin{aligned} AND(\bar{x}_1, x_2, \bar{x}_3, x_4, x_5) &= \text{minterm}(x_1, x_2, x_3, x_4, x_5 | 1, -1, 1, -1, -1) \\ &= \operatorname{sgn}(x_1 - x_2 + x_3 - x_4 - x_5 - 5). \end{aligned}$$

[1] If $\operatorname{sgn}(|X| - (n - 0.5))$ and $\operatorname{sgn}(|X| - 0.5)$ are used for AND and OR, we also get

$$AND(x_1, x_2, \dots, x_n) = \operatorname{sgn}(1 - n - \sum_{i=1}^{n} x_i), \quad OR(x_1, x_2, \dots, x_n) = \operatorname{sgn}(n - 1 - \sum_{i=1}^{n} x_i).$$

(c) Assume $b \neq a$, or we can not distinguish between these two Boolean values. If $x \in \{a, b\}$ is used to represent $x' \in \{0, 1\}$ via the mapping $0 \to a$ and $1 \to b$, we have

$$x' = \frac{x-b}{a-b}0 + \frac{x-a}{b-a}1 = \frac{x-a}{b-a}. \tag{2}$$

So the linear threshold representation of $f(X)$ for $X \in \{a, b\}^n$ is

$$f(X) = \operatorname{sgn}(w_0 + \sum_{i=1}^{n} w_i \frac{x_i - a}{b-a}) = \operatorname{sgn}(w_0 - \frac{a}{b-a}\sum_{i=1}^{n} w_i + \sum_{i=1}^{n} \frac{w_i}{b-a} x_i).$$

(d) Since the (2) is reversible, that is, $x = (b-a)x' + a$, we can also transform a linear threshold representation with $X \in \{a, b\}^n$ to a linear threshold representation with $X \in \{0, 1\}^n$. Since XOR is not a linear threshold function for $X \in \{0, 1\}^n$, there is no linear threshold representation for XOR for any a and b.

1.3 Simple Threshold Functions.

(a) From $|X| \leq n$ and definitions of AND, OR, and TH_i,

$$\begin{aligned} AND(X) &= \begin{cases} 1, & \text{if } |X| = n; \\ 0, & \text{otherwise.} \end{cases} = TH_n(X), \\ OR(X) &= \begin{cases} 1, & \text{if } |X| \geq 1; \\ 0, & \text{otherwise.} \end{cases} = TH_1(X). \end{aligned}$$

Thus $AND(X) \in TH$ and $OR(X) \in TH$.
However, $XOR(X) \notin TH$. Assume $XOR(X) = TH_i(X)$ for some i. By definition, $XOR(X) = 1$ when $|X| = 1$. So $TH_i(X)$ should also be 1, which implies $i \leq 1$. Thus for X with $|X| = 2$, $TH_i(X)$ is still 1, contradicting with $XOR(X) = 0$ when $|X| = 2$. We can then assume $XOR(X) = \overline{TH}_i(X)$. When $|X| = 0$, $XOR(X) = 0$. From the definition of $\overline{TH}_i$ we have $i = 0$. Then for $|X| = 1$, $\overline{TH}_i(X) = 0$, again contradicting with $XOR(X) = 1$ when $|X| = 1$. Thus we know $XOR(X) \notin TH$.

(b) Since $|X|$ is an integer, $|X| < i$ is equivalent to $|X| \leq i - 1$. Thus

$$\begin{aligned} TH_i(X) &= \begin{cases} 1, & \text{if } |X| \geq i; \\ 0, & \text{otherwise.} \end{cases} = \operatorname{sgn}(|X| - i) = \operatorname{sgn}(\sum_{j=1}^{n} x_j - i), \\ \overline{TH}_i(X) &= \begin{cases} 1, & \text{if } |X| < i; \\ 0, & \text{otherwise.} \end{cases} = \operatorname{sgn}(i - 1 - |X|) = \operatorname{sgn}(i - 1 - \sum_{j=1}^{n} x_j). \end{aligned}$$

So $TH_i \in LT_1$ and $\overline{TH}_i \in LT_1$, i.e., $TH = \{TH_i, \overline{TH}_i \,|0 \leq i \leq n\} \subseteq LT_1$.

1.4 Multilevel Threshold Functions. We assume $n \geq 2$ in this problem.

(a) Since $x_i \in \{0, 1, \ldots, q-1\}$, we have $\sum_{i=1}^{n} x_i \geq 0$ with equality iff $x_i = 0$ for all $1 \leq i \leq n$. So from the definition of OR_q, we get

$$OR_q(x_1, x_2, \ldots, x_n) = \begin{cases} 0, & \text{if } \sum_{i=1}^{n} x_i = 0; \\ 1, & \text{otherwise.} \end{cases}$$

Since $\sum_{i=1}^n x_i$ is also an integer, we further get

$$OR_q(x_1, x_2, \ldots, x_n) = \mathrm{sgn}(\sum_{i=1}^n x_i - 1).$$

Thus for any $q \geq 2$, $OR_q \in LT_1$.

(b) For $q = 2$, AND_q is the common AND and we already know $AND \in LT_1$.
For $q \geq 3$, suppose that $AND_q(x_1, x_2, \ldots, x_n) = \mathrm{sgn}(w_0 + \sum_{i=1}^n w_i x_i)$. Thus for below cases,

- $x_i = 1$ for all $1 \leq i \leq n$. $AND_q(x_1, x_2, \ldots, x_n) = 1$. So

$$w_0 + \sum_{i=1}^n w_i \geq 0. \tag{3}$$

- $x_i = 1$ for $i \neq j$, and $x_j = 0$. $AND_q(x_1, x_2, \ldots, x_n) = 0$. So

$$w_0 + \sum_{i=1}^n w_i - w_j < 0, \text{ or, } w_j > w_0 + \sum_{i=1}^n w_i \geq 0. \tag{4}$$

- $x_i = q - 1$ for $i \neq j$, and $x_j = 0$. $AND_q(x_1, x_2, \ldots, x_n) = 0$. So

$$w_0 + \sum_{i=1}^n (q-1)w_i - (q-1)w_j < 0. \tag{5}$$

Summing (5) over $j = 1, 2, \ldots, n$, we get

$$nw_0 + (n-1)(q-1)\sum_{i=1}^n w_i < 0. \tag{6}$$

Since $q \geq 3$ and $n \geq 2$, $(q-2)(n-1) \geq 1$. Thus with (4), we have

$$[(q-2)(n-1) - 1]\sum_{i=1}^n w_i \geq 0. \tag{7}$$

Subtracting (7) from (6) gives

$$n(w_0 + \sum_{i=1}^n w_i) < 0,$$

which contradicts with (3).
So only for $q = 2$, $AND_q \in LT_1$. □