Solutions to Problem Set 6.

Problem 1.
1. Linear program in standard form. There are two sets of constraints in the LP: m equality constraints $Ax = b$ and n inequality constraints $x \geq 0$. Therefore we shall introduce two vectors of Lagrange multipliers, calling them λ_e and λ_i, respectively. The Lagrangian will then become

$$\mathcal{L}(x, \lambda) = c^T x - \lambda_e^T (Ax - b) - \lambda_i^T x$$

The KKT conditions are as follows

$$\begin{aligned} \nabla_x \mathcal{L}(x, \lambda) = c - A^T \lambda_e - \lambda_i &= 0 \\ \lambda_i &\geq 0 \\ Ax = b, \quad x &\geq 0 \\ (\lambda_i)^T x &= 0 \end{aligned}$$

They have the familiar form of the LP optimality conditions introduced in the eariler lectures, if we set $y = \lambda_e$ and $s = \lambda_i$. Since for any solution $x \neq x^*$ we have $x^T s \geq 0$, it follows that $c^T x = x^T c = x^T (A^T y + s) = b^T y + x^T s \geq b^T y$, so x cannot have a lower objective value.
2. Trust region subproblem. The problem has the following form

$$\begin{aligned} \min \; & f_k + \nabla f_k^T p + \frac{1}{2} p^T B_k p \\ \text{s.t. } \; & \|p\|^2 = p^T p \leq \Delta_k^2 \quad \Leftrightarrow \quad -p^T p \geq -\Delta_k^2 \end{aligned}$$

with the Lagrangian

$$\mathcal{L}(p, \lambda) = f_k + \nabla f_k^T p + \frac{1}{2} p^T B_k p + \lambda (p^T p - \Delta_k^2)$$

The KKT conditions are

$$\begin{aligned} \nabla_p \mathcal{L} = \nabla f_k + B_k p + 2\lambda p &= 0 \\ p^T p \leq \Delta_k^2, \quad \lambda &\geq 0 \\ \lambda (p^T p - \Delta_k^2) &= 0 \end{aligned}$$

Problem 2.
Maximizing the perimeter is equivalent to minimizing its negative, so we can write the mathematical formulation of the problem in the "standard" form as follows

$$\begin{aligned} \min_{x,y} \; & -2x - 2y \\ \text{s.t. } \; & x^2 + y^2 - 4 = 0 \end{aligned}$$

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The Lagrangian is

$$\mathcal{L}(x, y, \lambda) = -2x - 2y - \lambda(x^2 + y^2 - 4)$$

and the optimality conditions imply

$$\frac{\partial \mathcal{L}}{\partial x} = -2 - 2\lambda^* x = 0, \quad \frac{\partial \mathcal{L}}{\partial y} = -2 - 2\lambda^* y = 0 \quad \Rightarrow \quad x^* = y^* = -\frac{1}{\lambda^*}$$

so that

$$(x^*)^2 + (y^*)^2 - 4 = -\frac{1}{(\lambda^*)^2} - \frac{1}{(\lambda^*)^2} - 4 = 0 \quad \Rightarrow \quad \lambda^* = -\frac{1}{\sqrt{2}}$$

and $(x^*, y^*) = (\sqrt{2}, \sqrt{2})$. Now, by definition,

$$p(\epsilon) = \min_{x^2+y^2=4+\epsilon} f(x(\epsilon), y(\epsilon)) = \min -2x - 2\sqrt{4 + \epsilon - x^2}$$

The latter function has a minimum at $x = \frac{1}{2}\sqrt{2(4+\epsilon)}$, as can be verified by standard calculus procedures, to yield

$$p(\epsilon) = -2\sqrt{2(4+\epsilon)} \quad \Rightarrow \quad \nabla p(0) = \frac{dp}{d\epsilon}(0) = -\frac{1}{\sqrt{2}} = \lambda^*$$

as required.

Problem 3.
Using the given formulas and the same idea as in problem 2 to convert a maximizing problem to a minimizing one, we can write the folowing mathematical formulation of the ice-cream melting problem:

$$\begin{gathered} \min \pi h^2 (h/3 - r) \\ \text{s.t. } 2\pi r h - \mathcal{A} = 0 \end{gathered}$$

where A is the given area of the spherical segment. The Lagrangian of the problem is

$$\mathcal{L}(r, h, \lambda) = \left(\frac{h}{3} - r\right) \pi h^2 - \lambda(2\pi r h - \mathcal{A})$$

The optimality conditions then imply

$$\frac{\partial \mathcal{L}}{\partial r} = -\pi h^2 - 2\pi \lambda h = 0, \quad \frac{\partial \mathcal{L}}{\partial h} = -2\pi r h + \pi h^2 - 2\pi \lambda r = 0 \quad \Rightarrow \quad h^* = -2\lambda^*, \; r^* = \frac{h^{*2}}{2\lambda^* + 2h^*}$$

so

$$r^* = h^* = \sqrt{\frac{\mathcal{A}}{2\pi}}, \quad \lambda^* = -\frac{1}{2}\sqrt{\frac{\mathcal{A}}{2\pi}}$$

Note that, as expected, maximum volume is attained when the height of the segment is equal to the radius of the sphere.

Problem 4.
The logarithmic barrier method invloves solving for unconstrained minimizers of

$$B(x_1, x_2, \mu) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - \mu \log(x_1 + x_2 - 1)$$

Setting the gradient to zero gives

$$x_1 - \frac{\mu}{x_1 + x_2 - 1} = 0, \quad x_2 - \frac{\mu}{x_1 + x_2 - 1} = 0$$

Thus

$$x_1 = x_2 = \frac{1 \pm \sqrt{1 + 8\mu}}{4}$$

Only the "+" solutions satisfy the constraint, so we have

$$x(\mu) = \begin{pmatrix} (1 + \sqrt{1 + 8\mu})/4 \\ (1 + \sqrt{1 + 8\mu})/4 \end{pmatrix}$$

$$\lambda(\mu) = \frac{\mu}{x_1 + x_2 - 1} = \frac{2\mu}{\sqrt{1 + 8\mu} - 1}$$

As $\mu \downarrow 0$, $x(\mu) \to x^* = (1/2, 1/2)^T$ and $\lambda(\mu) \to \lambda^* = \frac{1}{2}$. Now, the Hessian of the barrier function has the following form

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mu(x_1 + x_2 - 1)^{-2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

The attached plot demonstrates how the condition number of H grows as $\mu \downarrow 0$.

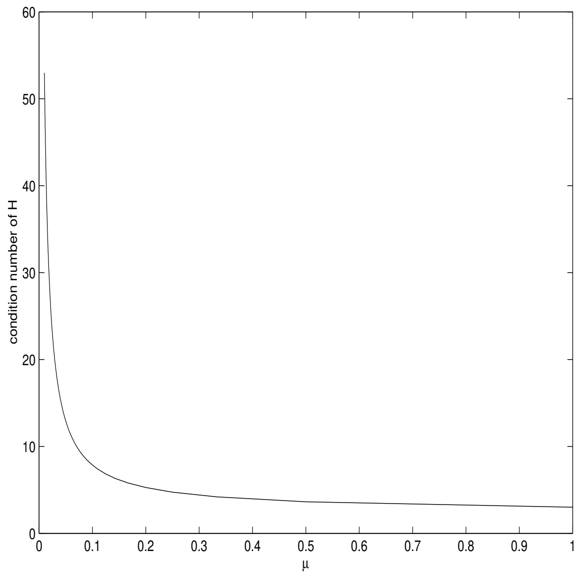

Figure 1: Condition number of the Hessian of the barrier function as $\mu \downarrow 0$

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Problem 5.

- Since $b^T x = 0$, the minimizer satisfies

  $$x^* = \arg\min \frac{1}{2} x^T Q x = 0$$

  since, by assumption, Q is positive definite for any x with $b^T x = 0$. To find λ^*, we write the first-order optimality condition

  $$\frac{\partial \mathcal{L}}{\partial x} = Qx + (\theta - \lambda) b = 0 \quad \Rightarrow \quad \lambda^* = \theta$$

- The augmented Lagrangian has the form

  $$\mathcal{L}_a = \frac{1}{2} x^T Q x + (\theta - \lambda) b^T x + \frac{a}{2} (b^T x)^2$$

  so setting its gradient to zero gives

  $$x = (\lambda_k - \theta - a(b^T x)) Q^{-1} b = \alpha Q^{-1} b$$

  for some scalar α. Multiplying the obtained identity by b^T on the left, we get, after some simplifications,

  $$\alpha = \frac{\lambda_k - \theta}{1 + a(b^T Q^{-1} b)} \quad \Rightarrow \quad x = \frac{\lambda_k - \theta}{1 + a b^T Q^{-1} b} Q^{-1} b$$

- Plugging $k = 1$ into the formula above verifies that x_1 is indeed given by the required formula. Similarly, $\lambda_1 = 0$ satisfies the corresponding formula for λ_k. Using induction,

  $$x_{k+1} = \frac{-\theta/(1 + a b^T Q^{-1} b)^k}{1 + a b^T Q^{-1} b} Q^{-1} b = -\frac{\theta Q^{-1} b}{(1 + a b^T Q {-} 1 b)^{k+1}}$$

  $$\begin{aligned} \lambda_{k+2} = \lambda_{k+1} - ac(x_{k+1}) = -\theta \left[\frac{1}{(1 + a b^T Q^{-1} b)^k} - 1 \right] - a b^T \left[\frac{\theta Q^{-1} b}{(1 + a b^T Q^{-1} b)^{k+1}} \right] &= \\ = -\theta \left[\frac{1}{(1 + a b^T Q^{-1} b)^{k+1}} - 1 \right] & \end{aligned}$$

  as required.

- Convergence is ensured if the denominator stays greater than 1 (otherwise, the sequence increases for a fixed a). Therefore convergence is guranteed when $a b^T Q^{-1} b > 0$. The larger the value of a, the faster the convergence.

Problem 6.

Dual Simplex Algorithm.

1. Find an initial basic feasible point and define $\mathcal{B}$ and $\mathcal{N}$, $s_N = c_N - N^T B^{-T} c_B \geq 0$, $s_B = 0$.
2. Set $k \leftarrow 1$.
3. Compute $x_B = B^{-1} b$.

4. If $x_B \geq 0$, then optimal solution is found.
5. Select $p \in \mathcal{B}$ with $x_p \leq 0$ as leaving index. Solve $Bt = A_p$ for t.
6. If $t \geq 0$, then the problem is unbounded.
7. Calculate
$$s_p^{k+1} = \max_{t_i<0, i \in \mathcal{N}} \frac{(s_N)_i}{t_i} \equiv \frac{(s_N)_q}{t_q}$$
8. Update $s_N^{k+1} = s_N^k - t s_p^{k+1}$.
9. Update $\mathcal{B}$ and $\mathcal{N}$ by removing q and adding p.
10. Set $k \leftarrow k+1$ and go to 3.

Now solve the given linear program. Introducing surplus variables, write the problem in standard form

$$\begin{aligned} \min 5x_1 + 4x_2 \\ \text{s.t. } 4x_1 + 3x_2 - x_3 = 10 \\ 3x_1 - 5x_2 - x_4 = 12 \\ x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Solve this using dual simplex method:

1st iteration

$$\mathcal{B} = \{x_3, x_4\},\ \mathcal{N} = \{x_1, x_2\}$$
$$c_B^T = (0,0),\ c_N^T = (5,4)$$
$$B^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad N = \begin{pmatrix} 4 & 3 \\ 3 & -5 \end{pmatrix}$$
$$x_B^{(1)} = (-10, -12)^T,\ y = (0,0)^T,\ s_N = (5,4)^T$$

x_4 is the leaving variable, and $t = (-3, 5)^T$. The ratios are $\{-5/3\}$, so x_1 is the entering variable.

2nd iteration

$$\mathcal{B} = \{x_3, x_1\},\ \mathcal{N} = \{x_4, x_2\}$$
$$c_B^T = (0,5),\ c_N^T = (0,4)$$
$$B^{-1} = \begin{pmatrix} -1 & 4/3 \\ 0 & 1/3 \end{pmatrix} \qquad N = \begin{pmatrix} 0 & 3 \\ -1 & -5 \end{pmatrix}$$
$$x_B^{(2)} = (6,4)^T,\ y = (0, 5/3)^T,\ s_N = (5/3, 37/3)^T$$

This is the optimal basis. The optimal primal solution is $x^* = (4,0)^T$; the optimal dual solution is $y^* = (0, 5/3)^T$; the optimal cost is $c_B^T x_B^* = 20$.

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Problem 7.
Write the relaxation LP in standard form, adding two slack variables:

$$\begin{aligned} \min & -x_2 \\ \text{s.t. } & 3x_1 + 2x_2 + x_3 = 6 \\ & -3x_1 + 2x_2 + x_4 = 0 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

We can use the dual simplex method to solve this linear program. The result is $x_B^{(1)} = (1, 3/2)^T$ with optimal cost $-x_2 = -3/2$. Because the solution is not integer, we generate cuts. The Gomory cut corresponding to row 2 is

$$\frac{1}{4}x_3 + \frac{1}{4}x_4 \geq \frac{1}{2}$$

which corresponds to the constraint $x_2 \leq 1$ in the original variables (using the constraints of the original relaxation LP). Adding a surplus variable x_5, we arrive at the following additional constraint

$$-\frac{1}{4}x_3 - \frac{1}{4}x_4 + x_5 = \frac{1}{2}$$

so the new LP is

$$\begin{aligned} \min & -x_2 \\ \text{s.t. } & 3x_1 + 2x_2 + x_3 = 6 \\ & -3x_1 + 2x_2 + x_4 = 0 \\ & -\frac{1}{4}x_3 - \frac{1}{4}x_4 + x_5 = \frac{1}{2} \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Solve it with dual simplex method to get a new optimal solution $x_B^{(2)} = (2/3, 1)^T$, which is still not an integer. The first row generates the Gomory cut

$$\frac{2}{3}x_4 + \frac{2}{3}x_5 \geq \frac{2}{3}$$

which is equivalent to $x_1 \geq x_2$. Introducing another surplus variable, we arrive at a new LP

$$\begin{aligned} \min & -x_2 \\ \text{s.t. } & 3x_1 + 2x_2 + x_3 = 6 \\ & -3x_1 + 2x_2 + x_4 = 0 \\ & -\frac{1}{4}x_3 - \frac{1}{4}x_4 + x_5 = \frac{1}{2} \\ & -\frac{2}{3}x_4 - \frac{2}{3}x_5 + x_6 = \frac{2}{3} \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned}$$

whose optimal solution, $x^* = (1, 1)^T$, is integer and solves the original ILP. The optimal value of the objective is $-x_2^* = -1$.

See the attached graphical representation of the constraints and cuts.

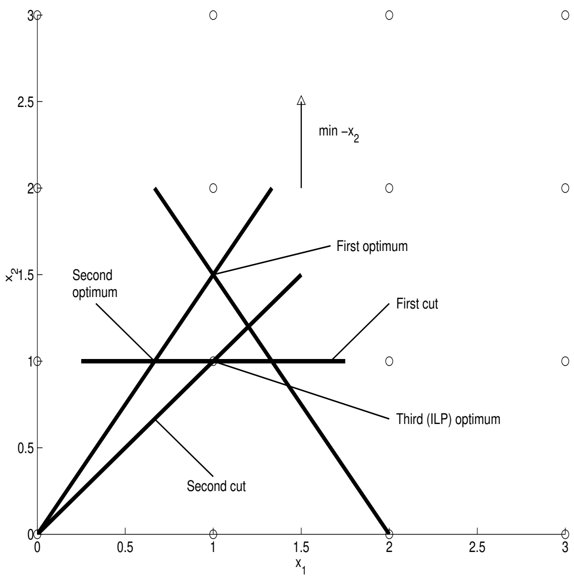

Figure 2: Solving the ILP using the cutting plane algorithm with Gomory cuts