ACM 113 **Final Exam** June 4, 2001

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Dear ACM113 Class Members,

The honor code is in effect. Please note that this is a closed-book exam to be completed in one 4 hour time period. Course notes and problem set solutions may be used during the exam but no other written material is allowed. If you have a question about the exam, feel free to contact me, but do not discuss the exam with anyone else. Completed exams should be returned to 313 Firestone by 9AM on Thursday, June 7.

With best wishes,

Niles.

1. The Baywatch Problem
A lifeguard sees a struggling swimmer and wants to minimize the amount of time it takes to run across the sand and swim to the victim. Given that the speeds over sand and through water are different (say v_1 and v_2), what path should the lifeguard follow to minimize the time to rescue? Draw a picture to illustrate your solution and simplify your answer so that it is described in terms of angles to the verticle axis (say θ_1 and θ_2).

2. Local Convergence of Trust Region Methods
Suppose $\nabla^2 f(x)$ is Lipschitz continuous and $\{x_k\}$ converges to a point x^* with $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ positive definite. Consider the trust region control algorithm with $B_k \equiv \nabla^2 f_k$ and steps p_k that yield at least the Cauchy point reduction ($m_k(p_k) \leq m_k(p_k^c)$) and are "asymptotically exact" (i.e. $||p_k - p_k^N|| = o(||p_k^N||)$) if $||p_k^N|| \leq \Delta_k/2$. Then the trust region bound Δ_k becomes inactive for k sufficiently large. We aim to show that the CG-Newton trust region method satisfies the conditions of this theorem and will therefore have local convergence properties determined by the choice of forcing sequence $\{n_k\}$ for the inner CG iteration.
a) Prove that the Cauchy point reduction in m_k satisfies

$$m_k(0) - m_k(p_k^c) \geq \frac{1}{2}||\nabla f_k|| \min\left(\Delta_k, \frac{||\nabla f_k||}{||\nabla^2 f_k||}\right),$$

and argue that the CG-Newton method yields at least this much reduction.
b) Prove that the CG-Newton method is asymptotically exact if the trust region constraint is inactive and $\eta_k \to 0$.

3. LP Sensitivity Analysis
Consider solutions to a linear program in standard form ($\min_x\ c^T x,\ Ax = b,\ x \geq 0,\ A \in R^{m\times n}$) that is modified in various ways. In each case, explain how to check whether the feasibility or optimility of the current basis has been affected. Under the assumption that they have not been affected, quantify the resulting change in the variables (x, y, s) and in the objective. Likewise, assuming that they have been affected, use your knowledge of the primal and dual simplex methods to suggest efficient approaches for restoring feasibility or optimality.

a) The vector b is perturbed by Δb.
b) A non-basic objective coefficient for x_{N_i} is perturbed by Δc_{N_i}.
c) The constraint coefficients for a non-basic variable x_{N_i} are perturbed by ΔN_i.
d) A new variable x_t is added with objective coefficient c_t and constraint coefficients $A_t = (a_{1,t}, ..., a_{m,t})^T$.

4. Augmented Lagrangian Methods
Consider the nonlinear program:

$$\begin{aligned} \min_x \ & \exp(x_1 x_2 x_3 x_4 x_5) \\ \text{s.t. } & x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - 10 = 0 \\ & x_2 x_3 - 5 x_4 x_5 = 0 \\ & x_1^3 + x_2^3 + 1 = 0 \end{aligned}$$

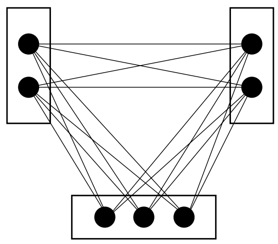
a) What are $L_a(x, \lambda)$ and $\nabla_x L_a(x, \lambda)$ for the problem?
b) Implement an augmented Lagrangian method using an unconstrained sub-iteration that is Hessian-free and at least superlinearly convergent. Describe your chosen approach via tersely commented source code.
c) Explain the expected performance trade-off between holding $a_k \geq \bar{a}$ fixed (for some $\bar{a} > 0$) and taking $a_k \to \infty$. Use the initial conditions $x_0 = (-2, 2, 2, -1, -1)^T$ and $\lambda_0 = (0, 0, 0)^T$ and report convergence of both x_k and λ_k. Do you observe the performance trade-off on this problem?

5. Protein Design is NP-complete
Consider the protein design "optimization problem": For p positions on the protein backbone, where position i has n_i amino acid side-chain alternatives, select the one side-chain at each position that minimizes the sum of the pairwise interaction energies between all positions*:

$$\min E_{\text{total}} = \sum_i \sum_{j, j<i} E(i_r, j_s).$$

Here, $E(i_r, j_s)$ is the pairwise interaction energy between side-chain r at position i and side-chain s at position j. A representative graph for the problem is shown below, where each box represents a backbone position, each circular node represents a possible side-chain alternative, and each edge connecting two nodes is weighted by the pairwise energy. In this picture, the goal is to choose the one node within each box that minimizes the sum of the three edge weights connecting the nodes.

a) What is the decision form of this problem (we'll call it PRODES)?
b) Show that PRODES is in NP.
c) Show that SATISFIABILITY is polynomially-transformable to PRODES and hence that PRODES is NP-complete.



*If you are uncomfortable with the biological terminology, you may also think of this as the "flight pricing problem": For a set of p cities, where city i contains n_i airports, choose the airport for each city that minimizes the cost of visiting every city from every other city.