ACM 113 Introduction to Optimization - Problem Set 4

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4.1 Let x and x' be any two feasible points for the linear program problem. Hence

$$Ax = Ax' = b, \quad x \geq 0, \text{ and } x' \geq 0.$$

Let $x^{(\lambda)} = \lambda x + (1-\lambda)x'$, where $0 \leq \lambda \leq 1$. Thus $x^{(\lambda)} \geq 0$, and

$$Ax^{(\lambda)} = \lambda Ax + (1-\lambda)Ax' = \lambda b + (1-\lambda)b = b.$$

So $x^{(\lambda)}$ is also a feasible point. Thus the feasible polytope is convex.

4.2 The claim in the problem is not true. For example,

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 6 \end{pmatrix},$$

where $m = n = 2$, and the row rank of A is $1 < m$. The feasible area of $Ax = b$ and $x \geq 0$ is a line segment from $(0,3)^T$ to $(3,0)^T$. Those two ends ($(0,3)^T$ and $(3,0)^T$) are basic feasible points.

4.3 The weak duality states $c^T x' \geq b^T y'$ for any primal feasible x' and any dual feasible y'. Thus we have $c^T x = b^T y \leq c^T x'$ and $b^T y = c^T x \geq b^T y'$. So x is primal optimal and y is dual optimal.

4.4 If the primal LP has a feasible point x, then from the weak duality, $b^T y' \leq c^T x$ for any dual feasible y'. That is, the dual LP objective is upper bounded. Symmetrically, if the dual LP has a feasible point y$'$, then $c^T x' \geq b^T y$ for any primal feasible x'. That is, the primal LP objective is lower bounded. Hence the claim in the problem holds.

4.5 Reorder the problem so that x_i' and x_i'' are the first two components of x. Assume there is a basic feasible solution $x = (x_i', x_i'', \dots)$ where $x_i' > 0$ and $x_i'' > 0$. Let ϵ be any positive number satisfying $\epsilon < x_i'$ and $\epsilon < x_i''$, and define

$$\begin{aligned} x^+ &= x + \epsilon(1,1,0,\dots,0) = (x_i' + \epsilon, x_i'' + \epsilon, \dots), \\ x^- &= x - \epsilon(1,1,0,\dots,0) = (x_i' - \epsilon, x_i'' - \epsilon, \dots). \end{aligned}$$

Since $x \geq 0$, we have $x^+, x^- \geq 0$ and $x^+ \neq x^-$. Since x_i' and x_i'' are introduced by the replacement $x_i = x_i' - x_i''$, we know as long as $x_i' - x_i''$ doesn't change and the other components of x remain the same, Ax will not change. So

$$Ax^+ = Ax^- = Ax = b.$$

Since we also have $x = \frac{1}{2}x^+ + \frac{1}{2}x^-$, by definition x can not be a basic feasible point. So our assumption is wrong and no basic feasible point can obtain both x_i' and x_i''.

4.6 Introduce slack variables s_1 and s_2. The standard form is

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b, \\ & x \geq 0, \end{aligned}$$

where $x = (x_1, x_2, x_3, x_4, s_1, s_2)^T$, $c = (-5, -7, -12, 1, 0, 0)^T$, $b = (38, 55)^T$, and

$$A = \begin{pmatrix} 2 & 3 & 2 & 1 & 1 & 0 \\ 3 & 2 & 4 & -1 & 0 & 1 \end{pmatrix}.$$

Starting with $x = (0, 0, 0, 0, 38, 55)^T$, by $y = B^{-T} c_B$, $s_N = c_N - N^T y$, $t = B^{-1} A_q$ and $x_B^{(k+1)} = x_B^{(k)} - x_q^{(k+1)} t$, we have table below:

k	basis	x_B	B	c_B	y	s_N	q	t	$x_q^{(k+1)}$
1	$\{5, 6\}$	$\begin{pmatrix} 38 \\ 55 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -5 \\ -7 \\ -12 \\ 1 \end{pmatrix}$	3	$\begin{pmatrix} 2 \\ 4 \end{pmatrix}$	$\frac{55}{4}$
2	$\{3, 5\}$	$\begin{pmatrix} \frac{55}{4} \\ \frac{21}{2} \end{pmatrix}$	$\begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix}$	$\begin{pmatrix} -12 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -3 \end{pmatrix}$	$\begin{pmatrix} 4 \\ -1 \\ -2 \\ 3 \end{pmatrix}$	4	$\begin{pmatrix} -\frac{1}{4} \\ \frac{3}{2} \end{pmatrix}$	7
3	$\{3, 4\}$	$\begin{pmatrix} \frac{31}{2} \\ 7 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}$	$\begin{pmatrix} -12 \\ 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{4}{3} \\ -\frac{7}{3} \end{pmatrix}$	$\begin{pmatrix} \frac{14}{3} \\ \frac{5}{3} \\ \frac{4}{3} \\ \frac{7}{3} \end{pmatrix}$	STOP		

We stop the simplex algorithm because $s_N \geq 0$. Thus the optimal point is $x = (0, 0, \frac{31}{2}, 7, 0, 0)^T$, and the minimum of the objective is -179.

4.7 Assume that the LP problem is not degenerate. Suppose at iteration k, variable x_1 left the basis and variable x_2 entered the basis. If at iteration $k+1$, variable x_1 re-entered the basis, there may be two cases for iteration $k+1$:

(a) x_2 left the basis. Thus the basis after iteration $k+1$ is the same as the basis before iteration k, which is impossible since the simplex method ensures that the objective decreases at each iteration.

(b) x_2 remained in the basis and an other variable in the basis at iteration k, say x_3, left the basis. Thus the basis for iteration $k+2$ can be formed by removing x_3 from and adding x_2 into the basis at iteration k. In another word, those two basis at iteration k and $k+2$ are adjacent.

Let B and N represent basic and non-basic variables for iteration k, thus ($q = 2$)

$$c^T x^{(k+1)} = c_B^T x_B^{(k)} + s_q x_q^{(k+1)},$$

where $x_q^{(k+1)} > 0$ is the maximal value subject to

$$x_B^{(k+1)} = x_B^{(k)} - B^{-1} A_q x_q^{(k+1)} \geq 0. \tag{1}$$

Since the objective after iteration $k+1$ is less than that after iteration $k+1$, i.e., $c^T x^{(k+2)} < c^T x^{(k+1)}$, and basis at iteration $k+1$ and $k+2$ are both adjacent to the basis at iteration k, we know

$$x_q^{(k+2)} > x_q^{(k+1)}$$

and $x_B^{(k+2)} = x_B^{(k)} - B^{-1} A_q x_q^{(k+2)}$. However, $x^{(k+2)}$ is a feasible solution, so $x_B^{(k+2)} \geq 0$, and $x_q^{(k+1)}$ is not the maximal value subject to (1). Thus this case is also impossible.

So, x_1 can not re-enter the basis at iteration $k+1$.

4.8 After adding the artificial variables, Phase I is supposed to solve

$$\begin{array}{ll} \text{minimize} & \tilde{c}^T \begin{pmatrix} x \\ a \end{pmatrix} \\ \text{subject to} & \begin{pmatrix} A & I_m \end{pmatrix} \begin{pmatrix} x \\ a \end{pmatrix} = Ax + a = b, \\ & x \geq 0,\ a \geq 0, \end{array}$$

where $\tilde{c} = (\underbrace{0, \ldots, 0}_{n}, \underbrace{1, \ldots, 1}_{m})^T$ and I_m is the $m \times m$ identity matrix. The dual conditions are

$$\begin{aligned} \begin{pmatrix} A^T \\ I_m \end{pmatrix} y + s &= \tilde{c}, && (2) \\ s &\geq 0, \\ \begin{pmatrix} x^T & a^T \end{pmatrix} s &= 0. && (3) \end{aligned}$$

Since the Phase I terminates at a basic feasible solution to the original problem, we have in the final Phase I basis, $x \geq 0$ and $a = 0$. Let B represent basic variables in x. From (3) we get $s_B = 0$. Then from (2), we have

$$B^T y = \tilde{c}_B - s_B = 0 - 0 = 0.$$

Since B is invertible, we know $y = 0$. Hence, (2) gives $s = \tilde{c}$, i.e., the reduced costs are zero for x and one for the artificial variables.