

Bounding $m_{\mathcal{H}}(N)$

To show: $m_{\mathcal{H}}(N)$ is polynomial

We show: $m_{\mathcal{H}}(N) \leq \dots \leq \dots \leq$ a polynomial

Key quantity:

$B(N, k)$: Maximum number of dichotomies on N points, with break point k

Recursive bound on $B(N, k)$

Consider the following table:

$$B(N, k) = \alpha + 2\beta$$

	# of rows	\mathbf{x}_1	\mathbf{x}_2	\dots	\mathbf{x}_{N-1}	\mathbf{x}_N
S_1	α	+1	+1	\dots	+1	+1
		-1	+1	\dots	+1	-1
		\vdots	\vdots	\vdots	\vdots	\vdots
		+1	-1	\dots	-1	-1
		-1	+1	\dots	-1	+1
S_2	β	+1	-1	\dots	+1	+1
		-1	-1	\dots	+1	+1
		\vdots	\vdots	\vdots	\vdots	\vdots
		+1	-1	\dots	+1	+1
		-1	-1	\dots	-1	+1
S_2^-	β	+1	-1	\dots	+1	-1
		-1	-1	\dots	+1	-1
		\vdots	\vdots	\vdots	\vdots	\vdots
		+1	-1	\dots	+1	-1
		-1	-1	\dots	-1	-1

Estimating α and β

Focus on $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{N-1}$ columns:

$$\alpha + \beta \leq B(N - 1, k)$$

	# of rows	\mathbf{x}_1	\mathbf{x}_2	\dots	\mathbf{x}_{N-1}	\mathbf{x}_N
S_1	α	+1	+1	\dots	+1	+1
		-1	+1	\dots	+1	-1
		\vdots	\vdots	\vdots	\vdots	\vdots
		+1	-1	\dots	-1	-1
		-1	+1	\dots	-1	+1
S_2	β	+1	-1	\dots	+1	+1
		-1	-1	\dots	+1	+1
		\vdots	\vdots	\vdots	\vdots	\vdots
		+1	-1	\dots	+1	+1
		-1	-1	\dots	-1	+1
S_2^-	β	+1	-1	\dots	+1	-1
		-1	-1	\dots	+1	-1
		\vdots	\vdots	\vdots	\vdots	\vdots
		+1	-1	\dots	+1	-1
		-1	-1	\dots	-1	-1

Estimating β by itself

Now, focus on the $S_2 = S_2^+ \cup S_2^-$ rows:

$$\beta \leq B(N - 1, k - 1)$$

	# of rows	\mathbf{x}_1	\mathbf{x}_2	...	\mathbf{x}_{N-1}	\mathbf{x}_N
S_1	α	+1	+1	...	+1	+1
		-1	+1	...	+1	-1
		⋮	⋮	⋮	⋮	⋮
		+1	-1	...	-1	-1
		-1	+1	...	-1	+1
S_2	β	+1	-1	...	+1	+1
		-1	-1	...	+1	+1
		⋮	⋮	⋮	⋮	⋮
		+1	-1	...	+1	+1
		-1	-1	...	-1	+1
S_2^-	β	+1	-1	...	+1	-1
		-1	-1	...	+1	-1
		⋮	⋮	⋮	⋮	⋮
		+1	-1	...	+1	-1
		-1	-1	...	-1	-1

Putting it together

$$B(N, k) = \alpha + 2\beta$$

$$\alpha + \beta \leq B(N - 1, k)$$

$$\beta \leq B(N - 1, k - 1)$$

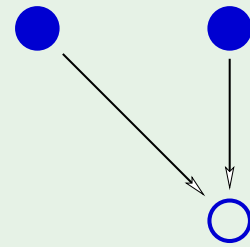
$$B(N, k) \leq$$

$$B(N - 1, k) + B(N - 1, k - 1)$$

	# of rows	\mathbf{x}_1	\mathbf{x}_2	\dots	\mathbf{x}_{N-1}	\mathbf{x}_N
S_1	α	+1	+1	\dots	+1	+1
		-1	+1	\dots	+1	-1
		\vdots	\vdots	\vdots	\vdots	\vdots
		+1	-1	\dots	-1	-1
		-1	+1	\dots	-1	+1
S_2	β	+1	-1	\dots	+1	+1
		-1	-1	\dots	+1	+1
		\vdots	\vdots	\vdots	\vdots	\vdots
		+1	-1	\dots	+1	+1
		-1	-1	\dots	-1	+1
S_2^-	β	+1	-1	\dots	+1	-1
		-1	-1	\dots	+1	-1
		\vdots	\vdots	\vdots	\vdots	\vdots
		+1	-1	\dots	+1	-1
		-1	-1	\dots	-1	-1

Numerical computation of $B(N, k)$ bound

$$B(N, k) \leq B(N - 1, k) + B(N - 1, k - 1)$$



		k						
		1	2	3	4	5	6	..
N	1	1	2	2	2	2	2	..
	2	1	3	4	4	4	4	..
	3	1	4	7	8	8	8	..
	4	1	5	11
	5	1	6	:	.			
	6	1	7	:		.		
	:	:	:	:			.	

Analytic solution for $B(N, k)$ bound

$$B(N, k) \leq B(N - 1, k) + B(N - 1, k - 1)$$

Theorem:

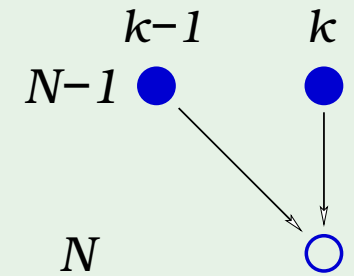
$$B(N, k) \leq \sum_{i=0}^{k-1} \binom{N}{i}$$

1. Boundary conditions: easy

		k						
		1	2	3	4	5	6	..
N	1	1	2	2	2	2	2	..
	2	1						
	3	1						
	4	1			●	●		
	5	1			↘	↓	○	
	6	1						
	⋮	⋮						

2. The induction step

$$\begin{aligned}\sum_{i=0}^{k-1} \binom{N}{i} &= \sum_{i=0}^{k-1} \binom{N-1}{i} + \sum_{i=0}^{k-2} \binom{N-1}{i} \text{ ?} \\ &= 1 + \sum_{i=1}^{k-1} \binom{N-1}{i} + \sum_{i=1}^{k-1} \binom{N-1}{i-1} \\ &= 1 + \sum_{i=1}^{k-1} \left[\binom{N-1}{i} + \binom{N-1}{i-1} \right] \\ &= 1 + \sum_{i=1}^{k-1} \binom{N}{i} = \sum_{i=0}^{k-1} \binom{N}{i} \checkmark\end{aligned}$$



It is polynomial!

For a given \mathcal{H} , the break point k is fixed

$$m_{\mathcal{H}}(N) \leq \underbrace{\sum_{i=0}^{k-1} \binom{N}{i}}_{\text{maximum power is } N^{k-1}}$$